

# Inversions of infinitely divisible distributions and conjugates of stochastic integral mappings

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The dual of an infinitely divisible distribution on  $\mathbb{R}^d$  without Gaussian part defined in Sato, ALEA **3** (2007), 67–110, is renamed to the inversion. Properties and characterization of the inversion are given. A stochastic integral mapping is a mapping  $\mu = \Phi_f \rho$  of  $\rho$  to  $\mu$  in the class of infinitely divisible distributions on  $\mathbb{R}^d$ , where  $\mu$  is the distribution of an improper stochastic integral of a nonrandom function  $f$  with respect to a Lévy process on  $\mathbb{R}^d$  with distribution  $\rho$  at time 1. The concept of the conjugate is introduced for a class of stochastic integral mappings and its close connection with the inversion is shown. The domains and ranges of the conjugates of three two-parameter families of stochastic integral mappings are described. Applications to the study of the limits of the ranges of iterations of stochastic integral mappings are made.

KEY WORDS: Infinitely divisible distribution; inversion; stochastic integral mapping; conjugate; monotone of order  $p$ ; increasing of order  $p$ ; class  $L_\infty$ .

## 1. INTRODUCTION

Let  $ID = ID(\mathbb{R}^d)$  be the class of infinitely divisible distributions on the  $d$ -dimensional Euclidean space  $\mathbb{R}^d$ . We use the Lévy–Khintchine representation of the characteristic function  $\hat{\mu}(z)$  of  $\mu \in ID$  in the form

$$\hat{\mu}(z) = \exp \left[ -\frac{1}{2} \langle z, A_\mu z \rangle + \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle 1_{\{|x| \leq 1\}}(x)) \nu_\mu(dx) + i\langle \gamma_\mu, z \rangle \right], \quad z \in \mathbb{R}^d,$$

where  $A_\mu$ ,  $\nu_\mu$ , and  $\gamma_\mu$  are the Gaussian covariance matrix, the Lévy measure, and the location parameter of  $\mu$ , respectively. A measure  $\nu$  on  $\mathbb{R}^d$  is the Lévy measure of some  $\mu \in ID$  if and only if  $\nu(\{0\}) = 0$  and  $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty$ . The class of the triplets  $(A_\mu, \nu_\mu, \gamma_\mu)$  represents the class  $ID$  one-to-one. If  $\int_{|x| \leq 1} |x| \nu_\mu(dx) < \infty$ , then  $\mu$  is said to have drift and  $\hat{\mu}(z)$  has expression

$$\hat{\mu}(z) = \exp \left[ -\frac{1}{2} \langle z, A_\mu z \rangle + \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1) \nu_\mu(dx) + i\langle \gamma_\mu^0, z \rangle \right],$$

where  $\gamma_\mu^0$  is called the drift of  $\mu$ . If  $\int_{|x| > 1} |x| \nu_\mu(dx) < \infty$ , then  $\mu$  has mean  $m_\mu$  and

$$\hat{\mu}(z) = \exp \left[ -\frac{1}{2} \langle z, A_\mu z \rangle + \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle) \nu_\mu(dx) + i\langle m_\mu, z \rangle \right].$$

Conversely, if  $\mu \in ID$  has mean, then  $\int_{|x| > 1} |x| \nu_\mu(dx) < \infty$ . These are basic facts; see Sato [21] for proofs. Let  $ID_0 = ID_0(\mathbb{R}^d)$  be the class of  $\mu \in ID$  with  $A_\mu = 0$ . For any

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subclass  $\mathfrak{C}$  of the class  $ID$ , let  $\mathfrak{C}_0$  denote  $\mathfrak{C} \cap ID_0$ . Sato [25], p. 85, introduced the dual  $\mu'$  of  $\mu \in ID_0$ . But the naming of the dual of  $\mu$  is usually used for the distribution  $\tilde{\mu}$  that satisfies  $\tilde{\mu}(B) = \mu(-B)$  for  $B \in \mathcal{B}(\mathbb{R}^d)$ , Borel sets in  $\mathbb{R}^d$ . So we call  $\mu'$  the inversion of  $\mu$  in this paper.

**Definition 1.1.** Let  $\mu \in ID_0$ . A distribution  $\mu' \in ID_0$  is the *inversion* of  $\mu$  if

$$\nu_{\mu'}(B) = \int_{\mathbb{R}^d \setminus \{0\}} 1_B(\iota(x)) |x|^2 \nu_{\mu}(dx), \quad B \in \mathcal{B}(\mathbb{R}^d) \quad (1.1)$$

$$\gamma_{\mu'} = -\gamma_{\mu} + \int_{|x|=1} x \nu_{\mu}(dx), \quad (1.2)$$

where  $\iota(x) = |x|^{-2}x$ , the geometric inversion of a point  $x \in \mathbb{R}^d \setminus \{0\}$ .

For any subclass  $\mathfrak{C}$  of  $ID_0$ , we will write  $\mathfrak{C}'$  for the class  $\{\mu' : \mu \in \mathfrak{C}\}$ . It is known that  $\mu'' = \mu$ , that  $\mu'$  has drift if and only if  $\mu$  has mean, and that  $\gamma_{\mu'}^0 = -m_{\mu}$ . Moreover, for  $0 < \alpha < 2$ ,  $\mu'$  is  $(2 - \alpha)$ -stable if and only if  $\mu$  is  $\alpha$ -stable;  $\mu'$  is strictly  $(2 - \alpha)$ -stable if and only if  $\mu$  is strictly  $\alpha$ -stable. These are shown in [25].

The main subject of our study is the analysis of improper stochastic integrals with respect to Lévy processes. Let  $\{X_t^{(\rho)} : t \geq 0\}$  be a Lévy process on  $\mathbb{R}^d$  such that  $\mathcal{L}(X_1^{(\rho)})$ , the distribution of  $X_1^{(\rho)}$ , equals  $\rho$ . Consider improper stochastic integrals with respect to  $\{X_t^{(\rho)}\}$  in two cases.

- (1) Let  $0 < c \leq \infty$  and let  $f(s)$  be a locally square-integrable function on  $[0, c)$  (that is,  $\int_0^q f(s)^2 ds < \infty$  for  $0 < q < c$ ). Then the stochastic integral  $\int_0^q f(s) dX_s^{(\rho)}$  is defined for  $0 < q < c$  for all  $\rho \in ID$ . We say that the improper stochastic integral  $\int_0^{c-} f(s) dX_s^{(\rho)}$  is definable if  $\int_0^q f(s) dX_s^{(\rho)}$  is convergent in probability (or almost surely, or in law, equivalently) as  $q \uparrow c$ . Let  $\Phi_f$  be the mapping from  $\rho$  to  $\Phi_f \rho = \mathcal{L}(\int_0^{c-} f(s) dX_s^{(\rho)})$ . Its domain  $\mathfrak{D}(\Phi_f)$  is the class  $\{\rho \in ID : \int_0^{c-} f(s) dX_s^{(\rho)} \text{ is definable}\}$ .
- (2) Let  $0 < c < \infty$  and let  $f(s)$  be a locally square-integrable function on  $(0, c]$ . Then  $\int_p^c f(s) dX_s^{(\rho)}$  is defined for  $0 < p < c$  for all  $\rho \in ID$ . We say that the improper stochastic integral  $\int_{0+}^c f(s) dX_s^{(\rho)}$  is definable if  $\int_p^c f(s) dX_s^{(\rho)}$  is convergent in probability (or almost surely, or in law, equivalently) as  $p \downarrow 0$ . Let  $\Phi_f$  be the mapping from  $\rho$  to  $\Phi_f \rho = \mathcal{L}(\int_{0+}^c f(s) dX_s^{(\rho)})$  with  $\mathfrak{D}(\Phi_f) = \{\rho \in ID : \int_{0+}^c f(s) dX_s^{(\rho)} \text{ is definable}\}$ .

The range  $\mathfrak{R}(\Phi_f) = \{\Phi_f \rho : \rho \in \mathfrak{D}(\Phi_f)\}$  is a subclass of  $ID$ . In any of the cases (1) and (2),  $\Phi_f$  is called a stochastic integral mapping as in [22, 23, 25, 26]. If  $c < \infty$  and  $\int_0^c f(s)^2 ds < \infty$ , then  $\int_0^{c-} f(s) dX_s^{(\rho)} = \int_{0+}^c f(s) dX_s^{(\rho)} = \int_0^c f(s) dX_s^{(\rho)}$  for all  $\rho \in ID$ . The analysis of  $\mathfrak{D}(\Phi_f)$  is rather complicated if  $\Phi_f = \Phi_{f_1}$  in case (1) with  $f_1(s) \asymp s^{-1}$  as  $s \rightarrow \infty$  (that is, there are positive constants  $c_1$  and  $c_2$  such that  $c_1 \leq f_1(s)s \leq c_2$  for all large  $s$ ) or if  $\Phi_f = \Phi_{f_2}$  in case (2) with  $f_2(s) \asymp s^{-1}$  as  $s \downarrow 0$ . This motivated us

to introduce the inversion (or the dual) in [25] in order to reduce the study of  $\Phi_{f_2}$  to that of  $\Phi_{f_1}$ . More generally, let  $f_1(s)$  and  $f_2(s)$  be locally square-integrable on  $[0, \infty)$  and on  $(0, c]$  with  $c < \infty$ , respectively. It is found in [25] that if the behavior of  $f_1(s)$  decreasing to 0 as  $s \rightarrow \infty$  and that of  $f_2(s)$  increasing to  $\infty$  as  $s \downarrow 0$  have some “relation,” then we can show that  $(\mathfrak{D}(\Phi_{f_1})_0)' = \mathfrak{D}(\Phi_{f_2})_0$ , using the inversion. This “relation” exists if, with  $0 < \alpha < 2$ ,  $f_1(s) \asymp s^{-1/\alpha}$  as  $s \rightarrow \infty$  and  $f_2(s) \asymp s^{-1/(2-\alpha)}$  as  $s \downarrow 0$ , or if  $\log(1/f_1(s)) \asymp s$  as  $s \rightarrow \infty$  and  $f_2(s) \asymp s^{-1/2}$  as  $s \downarrow 0$ . In these situations the study of  $\mathfrak{D}(\Phi_{f_1})$  for  $\int_0^{\infty-} f_1(s) dX_s^{(\rho)}$  is equivalent to that of  $\mathfrak{D}(\Phi_{f_2})$  for  $\int_{0+}^c f_2(s) dX_s^{(\rho)}$ . But the relationship of  $\mathfrak{R}(\Phi_{f_1})$  and  $\mathfrak{R}(\Phi_{f_2})$  is more delicate and it has not been studied so far.

The range  $\mathfrak{R}(\Phi_f)$  of a stochastic integral mapping first appeared with  $f(s) = e^{-s}$  in the representation of the class of selfdecomposable distributions by Wolfe [30] (see the historical notes in p.55 of [20]). Jurek [8] proposed the problem to find, for each limit theorem, a function  $f$  that describes the class of limit distributions for sequences of independent random variables as  $\mathfrak{R}(\Phi_f)$  (stochastic integral representation). Barndorff-Nielsen, Maejima, and Sato [4] described as  $\mathfrak{R}(\Phi_f)$  some well-known subclasses of  $ID$ . On the other hand, starting from the mappings  $\Phi_f$  for some explicitly given families of  $f$ , Sato [23] gave descriptions of  $\mathfrak{D}(\Phi_f)$  and  $\mathfrak{R}(\Phi_f)$ . The two lines are intertwined and many studies have been made ([5, 6, 8, 9, 10, 12, 13, 15, 25, 26] etc.); thus numerous connections between stochastic integral mappings and subclasses of  $ID$  have been found.

In this paper we continue to seek how to apply the inversion to the study of stochastic integral mappings. For the functions  $f_1(s)$  and  $f_2(s)$  above, we want to find in what situation we can say that  $(\mathfrak{R}(\Phi_{f_1})_0)' = \mathfrak{R}(\Phi_{f_2})_0$ . For this purpose we will introduce the notion of the conjugate of a stochastic integral mapping. Most of the stochastic integral mappings  $\Phi_f$  studied so far are such that  $f(s)$  is the inverse function of a function  $g(t)$  defined by a positive function  $h(u)$  as  $g(t) = \int_t^b h(u)du$  for  $t \in (a, b)$  with some  $a, b$  satisfying  $0 \leq a < b \leq \infty$ . This situation drew attention in [5] and Section 7 of [25]; in the terminology of [5] this is  $\Upsilon$ -transformation  $\Upsilon^\gamma$  corresponding to an absolutely continuous measure  $\gamma(du) = h(u)du$ . In defining the conjugate, we consider only this case (except in Section 5) and, writing  $g$  and  $f$  as  $g_h$  and  $f_h$ , let  $\Lambda_h$  denote  $\Phi_{f_h}$ . Under some condition, letting  $h^*(u) = h(u^{-1})u^{-4}$ , we will define the conjugate  $\Lambda_h^* = (\Lambda_h)^*$  of  $\Lambda_h$  as  $\Lambda_h^* = \Lambda_{h^*}$ . Then we will prove that  $(\Lambda_h^*)^* = \Lambda_h$  and that  $\rho \in \mathfrak{D}(\Lambda_h)_0$  and  $\mu = \Lambda_h \rho$  if and only if  $\rho' \in \mathfrak{D}(\Lambda_h^*)_0$  and  $\mu' = \Lambda_h^*(\rho')$ . Thus  $(\mathfrak{R}(\Lambda_h)_0)' = \mathfrak{R}(\Lambda_h^*)_0$ . Our next task is the study of  $\Lambda_h^*$ ,  $\mathfrak{D}(\Lambda_h^*)_0$ , and  $\mathfrak{R}(\Lambda_h^*)_0$  for several explicitly given mappings  $\Lambda_h$ . Specifically we will study  $\bar{\Phi}_{p,\alpha}$  and  $\Lambda_{q,\alpha}$  of Sato [26] and  $\Psi_{\alpha,\beta}$  of Maejima and Nakahara [13]; the definitions of these mappings will be given in Section 4.

The contents of the sections are as follows. Section 2 gives general properties and characterization of the inversion. Defining the dilation  $T_b\mu$ ,  $b > 0$ , of a measure  $\mu$  on  $\mathbb{R}^d$  as  $(T_b\mu)(B) = \int_{\mathbb{R}^d} 1_B(bx)\mu(dx)$ ,  $B \in \mathcal{B}(\mathbb{R}^d)$ , we find the relation between dilation and inversion, which enables us to treat easily semistable distributions under the action of inversion. In Section 3 we introduce conjugates of stochastic integral mappings and show their relations with inversion. Here not only the usual improper stochastic integrals but also their extension called essentially definable and their restriction called absolutely definable introduced in Sato's papers are treated. Their definitions are recalled in Section 3. These extension and restriction are more manageable than  $\Phi_f$  itself and give insight into the structure of  $\mathfrak{D}(\Phi_f)$  and  $\mathfrak{R}(\Phi_f)$ . Section 4 is devoted to explicit description of domains and ranges of the stochastic integral mappings  $\bar{\Phi}_{p,\alpha}$ ,  $\Lambda_{q,\alpha}$ , and  $\Psi_{\alpha,\beta}$  and their conjugates. Further, in the case of the conjugate of  $\Lambda_{1,\alpha}$ , a connection with the class  $L^{(\alpha)*}$  defined by a new kind of decomposability is given. The class  $L^{(\alpha)*}$  is shown to be the class of inversions of  $L^{(\alpha)}$ , where  $L^{(\alpha)}$  is the class of  $\alpha$ -selfdecomposable distributions in Jurek [9, 10] and Maejima and Ueda [16]. In particular,  $L^{(0)}$  is the class of selfdecomposable distributions. The functions  $f$  treated in Section 3 are positive and strictly decreasing. Section 5 gives some results similar to Section 3 for  $\Phi_f$  with a strictly decreasing function  $f$  taking positive and negative values both. Thus the class of inversions of type  $G$  distributions of Maejima and Rosiński [14] is treated. We make in Section 6 a study of  $\mathfrak{R}_\infty(\Lambda_h^*)$ , the limit of the nested classes  $\mathfrak{R}((\Lambda_h^*)^n)$ ,  $n = 1, 2, \dots$ . It is shown that  $\mathfrak{R}_\infty(\Lambda_h^*)_0 = (\mathfrak{R}_\infty(\Lambda_h)_0)'$ , which contributes to the study of the problem, treated in [15, 27] and others, concerning what classes can appear as  $\mathfrak{R}_\infty(\Phi_f)$  for general  $f$ .

For Lévy processes on  $\mathbb{R}^d$  the weak version of Shtatland's theorem [29] concerning  $\lim_{s \downarrow 0} s^{-1}X_s^{(\rho)}$  and the weak law of large numbers are obtained from each other through inversion. This remarkable application of the inversion will be given in another paper [28].

## 2. PROPERTIES AND CHARACTERIZATION OF INVERSIONS

First let us give a remark on the definition. In the two defining equations of the inversion of  $\mu \in ID_0$  in Definition 1.1, the expression of (1.2) depends on the choice of the integrand in the Lévy–Khintchine representation. If we define  $\gamma_\mu^\sharp$  for  $\mu \in ID_0$  by the representation

$$\widehat{\mu}(z) = \exp \left[ \int_{\mathbb{R}^d} \left( e^{i\langle z, x \rangle} - 1 - \frac{i\langle z, x \rangle}{1 + |x|^2} \right) \nu_\mu(dx) + i\langle \gamma_\mu^\sharp, z \rangle \right], \quad (2.1)$$

then (1.2) is written as  $\gamma_{\mu'}^\sharp = -\gamma_\mu^\sharp$ , since

$$\gamma_\mu^\sharp = \gamma_\mu - \int_{|x| \leq 1} \frac{|x|^2 x}{1 + |x|^2} \nu_\mu(dx) + \int_{|x| > 1} \frac{x}{1 + |x|^2} \nu_\mu(dx).$$

Thus our definition of the inversion of  $\mu$  is identical with the definition of the dual of  $\mu$  in Sato [25]. If we define  $\gamma_\mu^\sharp$  for  $\mu \in ID_0$  by

$$\widehat{\mu}(z) = \exp \left[ \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle 1_{\{|x| < 1\}}(x)) \nu_\mu(dx) + i\langle \gamma_\mu^\sharp, z \rangle \right], \quad (2.2)$$

then (1.2) is expressed as  $\gamma_{\mu'}^\sharp = -\gamma_\mu^\sharp - \int_{|x|=1} x \nu_\mu(dx)$ . If we define  $\gamma_\mu^\sharp$  for  $\mu \in ID_0$  by

$$\widehat{\mu}(z) = \exp \left[ \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle (1_{\{|x| < 1\}}(x) + \frac{1}{2} 1_{\{|x|=1\}}(x))) \nu_\mu(dx) + i\langle \gamma_\mu^\sharp, z \rangle \right], \quad (2.3)$$

then (1.2) is expressed as  $\gamma_{\mu'}^\sharp = -\gamma_\mu^\sharp$ . If we define  $\gamma_\mu^\sharp$  for  $\mu \in ID_0$  by

$$\widehat{\mu}(z) = \exp \left[ \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle c(x)) \nu_\mu(dx) + i\langle \gamma_\mu^\sharp, z \rangle \right] \quad (2.4)$$

with  $c(x) = 1_{\{|x| \leq 1\}}(x) + |x|^{-1} 1_{\{|x| > 1\}}(x)$  as in Rajput and Rosinski [19] and Kwapień and Woyczyński [11], then (1.2) is expressed as  $\gamma_{\mu'}^\sharp = -\gamma_\mu^\sharp + \int_{|x| \leq 1} |x| x \nu_\mu(dx) + \int_{|x| > 1} |x|^{-1} x \nu_\mu(dx)$ .

**Proposition 2.1.** *The inversion has the following properties.*

- (i) Any  $\mu \in ID_0$  has its inversion  $\mu' \in ID_0$ .
- (ii) The inversion of  $\mu'$  equals  $\mu$ , that is,  $\mu'' = \mu$ .
- (iii)  $\int_{|x| \leq 1} |x|^{2-\alpha} \nu_{\mu'}(dx) = \int_{|x| \geq 1} |x|^\alpha \nu_\mu(dx)$  for  $\alpha \in \mathbb{R}$ .
- (iv)  $\mu'$  has drift if and only if  $\mu$  has mean.
- (v) If  $\mu$  has mean, then  $\gamma_{\mu'}^0 = -m_\mu$ .
- (vi) If  $\mu$  and  $\mu_n$ ,  $n = 1, 2, \dots$ , are in  $ID_0$  and  $\mu_n \rightarrow \mu$ , then  $\mu'_n \rightarrow \mu'$ , where “ $\rightarrow$ ” denotes weak convergence.
- (vii)  $(\mu_1 * \mu_2)' = \mu'_1 * \mu'_2$  for  $\mu_1, \mu_2 \in ID_0$ .
- (viii)  $(\mu^s)' = (\mu')^s$  for  $\mu \in ID_0$  and  $s \geq 0$ , where  $\mu^s$  denotes the distribution with characteristic function  $(\widehat{\mu}(z))^s$ .
- (ix) If  $\mu = \delta_c$  with  $c \in \mathbb{R}^d$ , then  $\mu' = \delta_{-c}$ , where  $\delta_c$  denotes the  $\delta$ -distribution located at  $c \in \mathbb{R}^d$ .

Assertions (i)–(v) were already proved in [25], but here we repeat their proof for the convenience of readers.

*Proof of Proposition 2.1.* Given  $\mu \in ID_0$ , let  $\nu^\sharp(B)$ ,  $B \in \mathcal{B}(\mathbb{R}^d)$ , be the right-hand side of (1.1). Then  $\nu^\sharp(\{0\}) = 0$  and

$$\int_{\mathbb{R}^d} h(x) \nu^\sharp(dx) = \int_{\mathbb{R}^d \setminus \{0\}} h(\iota(x)) |x|^2 \nu_\mu(dx) \quad (2.5)$$

for any nonnegative measurable function  $h(x)$ . Thus  $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu^\sharp(dx) = \int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu_\mu(dx)$ . Hence, (i) is true. Moreover, it is readily proved that (2.5) is valid for any

$\mathbb{R}^d$ -valued measurable function  $h(x)$  on  $\mathbb{R}^d$  satisfying  $\int |h(x)|\nu^\sharp(dx) = \int |h(\iota(x))||x|^2 \nu_\mu(dx) < \infty$ . To see (ii), note that

$$\nu_{\mu''}(B) = \int_{\mathbb{R}^d \setminus \{0\}} 1_B(\iota(x))|x|^2 \nu_{\mu'}(dx) = \nu_\mu(B)$$

from (2.5) and that

$$\gamma_{\mu''} = -\gamma_{\mu'} + \int_{|x|=1} x \nu_{\mu'}(dx) = \gamma_\mu - \int_{|x|=1} x \nu_\mu(dx) + \int_{|x|=1} x \nu_{\mu'}(dx) = \gamma_\mu.$$

Assertion (iii) follows from (2.5); (iv) follows from (iii) with  $\alpha = 1$ . If  $\mu \in ID$  has drift, then  $\gamma_\mu^0 = \gamma_\mu - \int_{|x| \leq 1} x \nu_\mu(dx)$ . If  $\mu \in ID$  has mean, then  $m_\mu = \gamma_\mu + \int_{|x| > 1} x \nu_\mu(dx)$ . Hence we obtain (v) from (iv), noticing that

$$\gamma_{\mu'}^0 = \gamma_{\mu'} - \int_{|x| \leq 1} x \nu_{\mu'}(dx) = -\gamma_\mu + \int_{|x|=1} x \nu_\mu(dx) - \int_{|x| \geq 1} x \nu_\mu(dx) = -m_\mu.$$

To prove (vi) we use the expression (2.1) in order to apply Theorem 8.7 of [21]. We write  $f \in C_\sharp$  if  $f$  is a bounded continuous function from  $\mathbb{R}^d$  to  $\mathbb{R}$  vanishing on a neighborhood of 0. Let  $\mu$  and  $\mu_n$  be in  $ID_0$ . In order that  $\mu_n \rightarrow \mu$ , it is necessary and sufficient that  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f(x) \nu_{\mu_n}(dx) = \int_{\mathbb{R}^d} f(x) \nu_\mu(dx)$  for  $f \in C_\sharp$ ,  $\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \int_{|x| \leq \varepsilon} |x|^2 \nu_{\mu_n}(dx) = 0$ , and  $\lim_{n \rightarrow \infty} \gamma_{\mu_n}^\sharp = \gamma_\mu^\sharp$ . Now, assume that  $\mu_n \rightarrow \mu$ . Then, for  $f \in C_\sharp$ , we have

$$\int f(x) \nu_{\mu'_n}(dx) = \int_{|x| \leq \varepsilon} f(\iota(x))|x|^2 \nu_{\mu_n}(dx) + \int_{|x| > \varepsilon} f(\iota(x))|x|^2 \nu_{\mu_n}(dx) = I_1 + I_2,$$

where  $|I_1|$  is bounded by  $\|f\| \int_{|x| \leq \varepsilon} |x|^2 \nu_{\mu_n}(dx)$ , and  $I_2$  tends to  $\int_{|x| > \varepsilon} f(\iota(x))|x|^2 \nu_\mu(dx)$  as  $n \rightarrow \infty$  if  $\varepsilon$  is chosen to satisfy  $\int_{|x| = \varepsilon} \nu_\mu(dx) = 0$ . Hence, for  $f \in C_\sharp$ ,

$$\int f(x) \nu_{\mu'_n}(dx) \rightarrow \int f(\iota(x))|x|^2 \nu_\mu(dx) = \int f(x) \nu_{\mu'}(dx).$$

Moreover,  $\gamma_{\mu'_n}^\sharp = -\gamma_{\mu_n}^\sharp \rightarrow -\gamma_\mu^\sharp = \gamma_{\mu'}^\sharp$ , and  $\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \int_{|x| \leq \varepsilon} |x|^2 \nu_{\mu'_n}(dx) = 0$ , since  $\int_{|x| \leq \varepsilon} |x|^2 \nu'_n(dx) = \int_{|x| \geq 1/\varepsilon} \nu_{\mu_n}(dx)$ . Therefore  $\mu'_n \rightarrow \mu'$ . The converse follows from this by using (ii).

Since convolution induces addition in triplets, we have (vii). Since  $\mu^s$  has triplet  $(0, s\nu_\mu, s\gamma_\mu)$ , we have (viii). To see (ix), note that if  $\mu = \delta_c$ , then  $\widehat{\mu}(z) = e^{i\langle c, z \rangle}$ , so that  $\gamma_\mu = \gamma_\mu^0 = m_\mu = c$  and use (v).  $\square$

Let us give some characterization of the inversion.

**Proposition 2.2.** *Suppose that  $\mu \mapsto \mu^\sharp$  is a mapping from  $ID_0$  into  $ID_0$  such that, for some  $\alpha \in \mathbb{R}$ ,*

$$\nu_{\mu^\sharp}(B) = \int_{\mathbb{R}^d \setminus \{0\}} 1_B(\iota(x))|x|^\alpha \nu_\mu(dx) \text{ for } \mu \in ID_0 \text{ and } B \in \mathcal{B}(\mathbb{R}^d). \quad (2.6)$$

*Then  $\alpha = 2$ .*

*Proof.* Since  $\infty > \int_{|x| \leq 1} |x|^2 \nu_{\mu^\#}(dx) = \int_{|x| \geq 1} |x|^{\alpha-2} \nu_{\mu}(dx)$  for all  $\mu \in ID_0$ , we have  $\alpha \leq 2$ . Since  $\infty > \int_{|x| > 1} \nu_{\mu^\#}(dx) = \int_{|x| < 1} |x|^\alpha \nu_{\mu}(dx)$  for all  $\mu \in ID_0$ , we have  $\alpha \geq 2$ .  $\square$

In the following proposition let  $ID_{0c}$  denote the class of  $\mu \in ID_0$  with  $\nu_{\mu}$  having compact support in  $\mathbb{R}^d \setminus \{0\}$ , that is, satisfying  $\nu_{\mu}(\{|x| < a^{-1}\}) = \nu_{\mu}(\{|x| > a\}) = 0$  for some  $a > 1$ .

**Proposition 2.3.** *Suppose that  $\mu \mapsto \mu^\#$  is a mapping from  $ID_0$  into  $ID_0$  satisfying the following conditions:*

- (i) (2.6) is true with  $\alpha = 2$ ;
- (ii)  $\mu^{\#\#} = \mu$  for  $\mu \in ID_0$ ;
- (iii) there is  $k \in \mathbb{R}$  such that, for all  $\mu \in ID_{0c}$ ,  $\gamma_{\mu^\#}^0 = km_{\mu}$ ;
- (iv) If  $\mu$  and  $\mu_n$ ,  $n = 1, 2, \dots$ , are in  $ID_0$  and  $\mu_n \rightarrow \mu$ , then  $\mu_n^\# \rightarrow \mu^\#$ .

Then  $k = -1$  and  $\mu^\# = \mu'$  for  $\mu \in ID_0$ .

*Proof.* If  $\mu \in ID_{0c}$ , then  $\mu^\# \in ID_{0c}$  and  $\mu$  and  $\mu^\#$  have drift and mean. If  $\mu \in ID_{0c}$ , then the identity  $\gamma_{\mu^\#}^0 = km_{\mu}$  is written as

$$\gamma_{\mu^\#} - \int_{|x| \leq 1} x \nu_{\mu^\#} = k \left( \gamma_{\mu} + \int_{|x| > 1} x \nu_{\mu}(dx) \right),$$

that is,

$$\gamma_{\mu^\#} = k\gamma_{\mu} + (k+1) \int_{|x| > 1} x \nu_{\mu}(dx) + \int_{|x|=1} x \nu_{\mu}(dx),$$

since (2.5) is true with  $\nu_{\mu^\#}$  in place of  $\nu^\#$ . Hence, if  $\mu \in ID_{0c}$ , then

$$\gamma_{\mu^{\#\#}} = k^2\gamma_{\mu} + k(k+1) \int_{|x| > 1} x \nu_{\mu}(dx) + (k+1) \int_{|x|=1} x \nu_{\mu}(dx) + (k+1) \int_{|x| < 1} x \nu_{\mu}(dx),$$

which combined with condition (ii) says that

$$(1 - k^2)\gamma_{\mu} = k(k+1) \int_{|x| > 1} x \nu_{\mu}(dx) + (k+1) \int_{|x| \leq 1} x \nu_{\mu}(dx).$$

This is absurd if  $k \neq -1$ . Indeed, if  $k^2 \neq 1$ , then this would mean that

$$\gamma_{\mu} = \frac{1+k}{1-k^2} \left( k \int_{|x| > 1} x \nu_{\mu}(dx) + \int_{|x| \leq 1} x \nu_{\mu}(dx) \right)$$

for all  $\mu \in ID_{0c}$ ; if  $k = 1$ , then this would mean that  $0 = 2 \int_{\mathbb{R}^d} x \nu_{\mu}(dx)$  for all  $\mu \in ID_{0c}$ . Therefore  $k = -1$ . Hence  $\gamma_{\mu^\#} - \int_{|x| \leq 1} x \nu_{\mu^\#}(dx) = -\gamma_{\mu} - \int_{|x| > 1} x \nu_{\mu}(dx)$ , that is,  $\gamma_{\mu^\#} = \gamma_{\mu'}$  for all  $\mu \in ID_{0c}$ . Hence  $\mu^\# = \mu'$  for all  $\mu \in ID_{0c}$ . Approximating a general  $\mu \in ID_0$  by  $\mu_n \in ID_{0c}$  and using condition (iv) together with Proposition 2.1 (vi), we obtain  $\mu^\# = \mu'$  for all  $\mu \in ID_0$ .  $\square$

The dilation  $T_b\mu$  of a measure  $\mu$  on  $\mathbb{R}^d$  is defined in Section 1.

**Proposition 2.4.** *Let  $b > 0$ . Then  $(T_b\mu)' = (T_{b^{-1}}(\mu'))^{b^2}$  for  $\mu \in ID_0$ .*

*Proof.* We have

$$\nu_{T_b\mu} = T_b\nu_\mu \quad \text{and} \quad \gamma_{T_b\mu} = b\gamma_\mu \begin{cases} +b \int_{1 < |x| \leq b^{-1}} x\nu_\mu(dx) & \text{if } b < 1 \\ -b \int_{b^{-1} < |x| \leq 1} x\nu_\mu(dx) & \text{if } b > 1. \end{cases} \quad (2.7)$$

Assume that  $b > 1$ . Then

$$\begin{aligned} \nu_{(T_b\mu)'}(B) &= b^2 \int 1_{bB}(\iota(x))|x|^2 \nu_\mu(dx) = b^2 \nu_{\mu'}(bB) = b^2 (T_{b^{-1}}(\nu_{\mu'}))(B), \\ \gamma_{(T_b\mu)'} &= -\gamma_{T_b\mu} + b \int_{|x|=b^{-1}} x\nu_\mu(dx) = -b\gamma_\mu + b \int_{b^{-1} \leq |x| \leq 1} x\nu_\mu(dx) \\ &= b\gamma_{\mu'} + b \int_{1 < |x| \leq b} x\nu_{\mu'}(dx) = b^2 \gamma_{T_{b^{-1}}(\mu')}. \end{aligned}$$

This proves the assertion for  $b > 1$ . This result and Proposition 2.1 (viii) yield the assertion for  $0 < b < 1$ . It is trivial for  $b = 1$ .  $\square$

Let  $0 < \alpha \leq 2$ ,  $b > 1$ , and  $\mu \in ID$ . We say that  $\mu$  is  $\alpha$ -semistable [resp. strictly  $\alpha$ -semistable] with a span  $b$  if  $\mu^{b^\alpha} = (T_b\mu) * \delta_c$  for some  $c \in \mathbb{R}^d$  [resp.  $\mu^{b^\alpha} = T_b\mu$ ]. We say that  $\mu$  is  $\alpha$ -stable [resp. strictly  $\alpha$ -stable] if, for all  $b > 1$ ,  $\mu$  is  $\alpha$ -semistable [resp. strictly  $\alpha$ -semistable] with a span  $b$ . Thus any trivial distribution (that is,  $\delta$ -distribution) is  $\alpha$ -stable for  $0 < \alpha \leq 2$ .

The following theorem gives further remarkable properties of the inversion. If  $\mu$  is  $\alpha$ -semistable with  $0 < \alpha < 2$ , then  $\mu \in ID_0$ . Assertions (iii) and (iv) were shown in [25], but we will give a new proof.

**Theorem 2.5.** *Let  $0 < \alpha < 2$ ,  $b > 1$ , and  $\mu \in ID_0$ .*

- (i)  $\mu'$  is  $(2 - \alpha)$ -semistable with a span  $b$  if and only if  $\mu$  is  $\alpha$ -semistable with a span  $b$ .
- (ii)  $\mu'$  is strictly  $(2 - \alpha)$ -semistable with a span  $b$  if and only if  $\mu$  is strictly  $\alpha$ -semistable with a span  $b$ .
- (iii)  $\mu'$  is  $(2 - \alpha)$ -stable if and only if  $\mu$  is  $\alpha$ -stable.
- (iv)  $\mu'$  is strictly  $(2 - \alpha)$ -stable if and only if  $\mu$  is strictly  $\alpha$ -stable.

*Proof.* In general we have, for  $b, b_1, b_2 > 0$ ,  $T_b(\mu_1 * \mu_2) = (T_b\mu_1) * (T_b\mu_2)$ ,  $T_b(\mu^s) = (T_b\mu)^s$ ,  $T_{b_2}(T_{b_1}\mu) = T_{b_2b_1}\mu$ , and  $T_b(\delta_c) = \delta_{bc}$ . Let us prove assertion (i). Let  $0 < \alpha < 2$  and  $b > 1$ . Assume that  $\mu$  is  $\alpha$ -semistable with a span  $b$ . Then  $\mu^{b^\alpha} = (T_b\mu) * \delta_c$ , and hence  $(T_{b^{-1}}\mu)^{b^\alpha} = \mu * \delta_{b^{-1}c}$ , that is,  $\mu = (T_{b^{-1}}\mu)^{b^\alpha} * \delta_{-b^{-1}c}$ . This gives  $\mu^{b^{-\alpha}} = (T_{b^{-1}}\mu) * \delta_{-b^{-\alpha-1}c}$ . Now go to inversions and use Propositions 2.1 and 2.4. Then  $(\mu')^{b^{-\alpha}} = (T_b(\mu'))^{b^{-2}} * \delta_{b^{-\alpha-1}c}$ . Hence  $(\mu')^{b^{2-\alpha}} = (T_b(\mu')) * \delta_{b^{1-\alpha}c}$ , and  $\mu'$  is  $(2 - \alpha)$ -semistable with a span  $b$ . The converse is also proved from this, since  $\mu'' = \mu$ . Thus (i) is true. Assertion (ii) is shown by letting  $c = 0$  in the argument above. Assertions (iii) and (iv) are automatic from (i) and (ii). Another proof of this theorem can



be given by using the characterization of Lévy measures of (strictly) semistable and stable distributions in [21].  $\square$

Any  $\sigma$ -finite measure  $\nu$  on  $\mathbb{R}^d$  with  $\nu(\{0\}) = 0$  has two decompositions. Let  $S = \{\xi \in \mathbb{R}^d: |\xi| = 1\}$ , the unit sphere in  $\mathbb{R}^d$ , and  $\mathbb{R}_+^\circ = (0, \infty)$ . (1) There are a  $\sigma$ -finite measure  $\lambda$  on  $S$  with  $\lambda(S) \geq 0$  and a measurable family  $\{\nu_\xi: \xi \in S\}$  of  $\sigma$ -finite measures on  $\mathbb{R}_+^\circ$  with  $\nu_\xi(\mathbb{R}_+^\circ) > 0$  such that  $\nu(B) = \int_S \lambda(d\xi) \int_{\mathbb{R}_+^\circ} 1_B(r\xi) \nu_\xi(dr)$ ,  $B \in \mathcal{B}(\mathbb{R}^d)$ . The pair  $(\lambda(d\xi), \nu_\xi(dr))$  is called a radial decomposition of  $\nu$ . It is unique in the sense that, if  $(\lambda^1(d\xi), \nu_\xi^1(dr))$  and  $(\lambda^2(d\xi), \nu_\xi^2(dr))$  are both radial decompositions of  $\nu$ , then, for some positive, finite, measurable function  $c(\xi)$  on  $S$ , we have  $c(\xi)\lambda^2(d\xi) = \lambda^1(d\xi)$  and  $\nu_\xi^2(dr) = c(\xi)\nu_\xi^1(dr)$  for  $\lambda^1$ -a.e.  $\xi \in S$ . (2) There are a  $\sigma$ -finite measure  $\bar{\nu}$  on  $\mathbb{R}_+^\circ$  with  $\bar{\nu}(\mathbb{R}_+^\circ) \geq 0$  and a measurable family  $\{\lambda_r: r \in \mathbb{R}_+^\circ\}$  of  $\sigma$ -finite measures on  $S$  with  $\lambda_r(S) > 0$  such that  $\nu(B) = \int_{\mathbb{R}_+^\circ} \bar{\nu}(dr) \int_S 1_B(r\xi) \lambda_r(d\xi)$ ,  $B \in \mathcal{B}(\mathbb{R}^d)$ . The pair  $(\bar{\nu}(dr), \lambda_r(d\xi))$  is called a spherical decomposition of  $\nu$ . It is unique in a sense similar to in (1). See Sato [26], pp. 27–28 for details.

**Example 2.6.** Suppose that  $\mu$  is 1-stable. Then  $\mu' = \mu * \delta_{-2\gamma_\mu}$ . Indeed,  $\nu_\mu$  has a radial decomposition  $(\lambda(d\xi), r^{-2}dr)$  and hence

$$\nu_{\mu'}(B) = \int_S \lambda(d\xi) \int_{\mathbb{R}_+^\circ} 1_B(r^{-1}\xi) dr = \int_S \lambda(d\xi) \int_{\mathbb{R}_+^\circ} 1_B(r\xi) r^{-2} dr = \nu_\mu(B)$$

and  $\gamma_{\mu'} = -\gamma_\mu$ . Thus

$$\widehat{\mu'}(z) = \exp \left[ \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle 1_{\{|x| \leq 1\}}(x)) \nu_\mu(dx) - i\langle \gamma_\mu, z \rangle \right].$$

If  $\gamma_\mu = 0$ , then  $\mu$  is self-inversion, that is,  $\mu' = \mu$ . If  $\nu_\mu \neq 0$ , then  $\lambda \neq 0$  and  $(r^{-2}dr, \lambda)$  is a spherical decomposition of  $\nu_\mu$  at the same time.

**Example 2.7.** Let  $d = 1$ . Let  $\mu$  be Poisson distribution with mean  $m > 0$ . Then  $\nu_\mu = m\delta_1$  and hence  $\nu_{\mu'} = m\delta_1 = \nu_\mu$ . Thus  $\mu'$  is translated Poisson distribution. Since  $\mu$  has drift 0 and mean  $m$ ,  $\mu'$  has mean 0 and drift  $-m$  and  $\mu' = \mu * \delta_{-m}$ . Note that  $\mu * \delta_{-m/2}$  is self-inversion.

### 3. CONJUGATES OF STOCHASTIC INTEGRAL MAPPINGS

We introduce Condition (C) on a function  $h$  and define the conjugate of a stochastic integral mapping associated with a function  $h$  satisfying this condition. Then we give main results on the connection between the conjugate and the inversion.

**Definition 3.1.** A function  $h$  is said to satisfy *Condition (C)* if there are  $a_h$  and  $b_h$  with  $0 \leq a_h < b_h \leq \infty$  such that  $h$  is defined on  $(a_h, b_h)$ , positive, and measurable,

and at least one of the following is true:

$$\int_{a_h}^{b_h} h(u)u^2 du < \infty, \quad (3.1)$$

$$\int_{a_h}^{b_h} h(u) du < \infty. \quad (3.2)$$

A function  $h$  satisfying Condition (C) with (3.1) [resp. (3.2)] is said to satisfy (C<sub>1</sub>) [resp. (C<sub>2</sub>)].

**Definition 3.2.** Let  $h$  be a function satisfying Condition (C). Define a function  $h^*$  as  $a_{h^*} = 1/b_h$ ,  $b_{h^*} = 1/a_h$  (letting  $1/0 = \infty$  and  $1/\infty = 0$ ), and

$$h^*(u) = h(u^{-1})u^{-4}, \quad u \in (a_{h^*}, b_{h^*}). \quad (3.3)$$

**Proposition 3.3.** *If  $h$  satisfies Condition (C), then  $h^*$  satisfies Condition (C). If  $h$  satisfies (C<sub>1</sub>), then  $h^*$  satisfies (C<sub>2</sub>). If  $h$  satisfies (C<sub>2</sub>), then  $h^*$  satisfies (C<sub>1</sub>). Moreover,  $(h^*)^* = h$ .*

*Proof.* Notice that

$$\begin{aligned} \int_{a_{h^*}}^{b_{h^*}} h^*(u)u^2 du &= \int_{1/b_h}^{1/a_h} h(u^{-1})u^{-4}u^2 du = \int_{a_h}^{b_h} h(v)dv, \\ \int_{a_{h^*}}^{b_{h^*}} h^*(u) du &= \int_{1/b_h}^{1/a_h} h(u^{-1})u^{-4} du = \int_{a_h}^{b_h} h(v)v^2 dv. \end{aligned}$$

Then the assertions on  $h^*$  follow from the properties of  $h$ . The relation  $(h^*)^* = h$  is obvious.  $\square$

For each function  $h(u)$  satisfying Condition (C), let

$$g_h(t) = \int_t^{b_h} h(u) du, \quad t \in (a_h, b_h). \quad (3.4)$$

Then  $g_h(t)$  is a strictly decreasing, continuous function with  $g_h(b_h-) = 0$ . Let  $c_h = g_h(a_h+)$ . Define  $f_h(s)$  as

$$s = g_h(t) \text{ with } a_h < t < b_h \quad \Leftrightarrow \quad t = f_h(s) \text{ with } 0 < s < c_h.$$

Then  $f_h(s)$  is a strictly decreasing, continuous function with  $f_h(0+) = b_h$  and  $f_h(c_h-) = a_h$ , and

$$\int_u^{c_h} f_h(s)^2 ds < \infty, \quad u \in (0, c_h), \quad (3.5)$$

since

$$\int_u^{c_h} f_h(s)^2 ds = \int_{f_h(u)}^{a_h} t^2 dg_h(t) = \int_{a_h}^{f_h(u)} h(t)t^2 dt.$$

We have

$$\int_0^{c_h} f_h(s)^2 ds < \infty \quad \text{if } h \text{ satisfies (C}_1\text{)}, \quad (3.6)$$

$$c_h < \infty \quad \text{if } h \text{ satisfies (C}_2\text{)}. \quad (3.7)$$

Define a stochastic integral mapping  $\Phi_{f_h}$  as  $\Phi_f$  in Section 1 with  $f = f_h$ . Indeed, we have, for  $\rho \in \mathfrak{D}(\Phi_{f_h})$ ,

$$\begin{aligned} \Phi_{f_h}\rho &= \mathcal{L} \left( \int_0^{c_h^-} f_h(s) dX_s^{(\rho)} \right) && \text{if } h \text{ satisfies (C}_1\text{)}, \\ \Phi_{f_h}\rho &= \mathcal{L} \left( \int_{0+}^{c_h} f_h(s) dX_s^{(\rho)} \right) && \text{if } h \text{ satisfies (C}_2\text{)}, \end{aligned}$$

and

$$\Phi_{f_h}\rho = \mathcal{L} \left( \int_0^{c_h} f_h(s) dX_s^{(\rho)} \right) \quad \text{if } h \text{ satisfies (C}_1\text{) and (C}_2\text{)}.$$

**Definition 3.4.** If  $h$  is a function satisfying Condition (C), then  $\Phi_{f_h}$  is written as  $\Lambda_h$ . We call the stochastic integral mapping  $\Lambda_{h^*}$  the *conjugate* of  $\Lambda_h$  and write  $\Lambda_{h^*}$  as  $\Lambda_h^*$ . Thus  $\Lambda_h^* = \Lambda_{h^*} = \Phi_{f_{h^*}}$ .

**Proposition 3.5.** *The conjugate of  $\Lambda_h^*$  coincides with  $\Lambda_h$ .*

*Proof.* This is a direct consequence of Proposition 3.3.  $\square$

In general, given a function  $h$  satisfying Condition (C), we write  $a, b, c, g, f, a_*, b_*, c_*, g_*$ , and  $f_*$  for  $a_h, b_h, c_h, g_h, f_h, a_{h^*}, b_{h^*}, c_{h^*}, g_{h^*}$ , and  $f_{h^*}$ , respectively, if no confusion arises.

In the study of a stochastic integral mapping  $\Phi_f$  it is important to use some extension and some restriction of  $\Phi_f$ , because they are more manageable than  $\Phi_f$  itself and give information on the structure of the domain and the range. Suppose that  $h$  satisfies (C<sub>1</sub>) [resp. (C<sub>2</sub>)]. A distribution  $\rho \in ID$  is in  $\mathfrak{D}(\Lambda_h)$  if and only if  $\int_0^q \log \widehat{\rho}(f_h(s)z) ds$  [resp.  $\int_p^{c_h} \log \widehat{\rho}(f_h(s)z) ds$ ] is convergent as  $q \uparrow c_h$  [resp.  $p \downarrow 0$ ] for every  $z \in \mathbb{R}^d$  ([23], p. 51). We say that  $\Lambda_h \rho$  is absolutely definable if  $\int_0^{c_h} |\log \widehat{\rho}(f_h(s)z)| ds < \infty$  for every  $z \in \mathbb{R}^d$ . We say that  $\Lambda_h \rho$  is essentially definable if, for some  $\mathbb{R}^d$ -valued function  $k$  on  $[0, c_h)$  [resp.  $(0, c_h]$ ] and some  $\mathbb{R}^d$ -valued random variable  $Y$ ,  $\int_0^q f_h(s) dX_s^{(\rho)} - k(q)$  [resp.  $\int_p^{c_h} f_h(s) dX_s^{(\rho)} - k(p)$ ] converges to  $Y$  in probability as  $q \uparrow c_h$  [resp.  $p \downarrow 0$ ]. Define

$$\mathfrak{D}^0(\Lambda_h) = \{\rho \in ID : \Lambda_h \rho \text{ is absolutely definable}\},$$

$$\mathfrak{D}^e(\Lambda_h) = \{\rho \in ID : \Lambda_h \rho \text{ is essentially definable}\},$$

$$\mathfrak{R}^0(\Lambda_h) = \{\mu = \Lambda_h \rho : \rho \in \mathfrak{D}^0(\Lambda_h)\},$$

$$\mathfrak{R}^e(\Lambda_h) = \{\mu = \mathcal{L}(Y) : \rho \in \mathfrak{D}^e(\Lambda_h) \text{ and all } k \text{ and } Y \text{ that can be chosen}\}$$

in the definition of essential definability of  $\Lambda_h \rho$ .

Then  $\mathfrak{D}^0(\Lambda_h) \subset \mathfrak{D}(\Lambda_h) \subset \mathfrak{D}^e(\Lambda_h)$  and  $\mathfrak{R}^0(\Lambda_h) \subset \mathfrak{R}(\Lambda_h) \subset \mathfrak{R}^e(\Lambda_h)$ . The condition for  $\rho$  or  $\mu$  in  $ID$  to belong to these classes can be described in terms of their triplets (see [23, 25, 26]).

**Theorem 3.6.** *Let  $h$  be a function satisfying Condition (C). Consider  $\Lambda_h$  and its conjugate  $\Lambda_h^*$ . Let  $\rho \in ID_0$ . Then*

$$\rho \in \mathfrak{D}(\Lambda_h) \quad \text{and} \quad \Lambda_h \rho = \mu \quad (3.8)$$

*if and only if*

$$\rho' \in \mathfrak{D}(\Lambda_h^*) \quad \text{and} \quad \Lambda_h^* \rho' = \mu' \quad (3.9)$$

*Furthermore,*

$$\mathfrak{D}(\Lambda_h^*)_0 = (\mathfrak{D}(\Lambda_h)_0)', \quad (3.10)$$

$$\mathfrak{D}^e(\Lambda_h^*)_0 = (\mathfrak{D}^e(\Lambda_h)_0)', \quad (3.11)$$

$$\mathfrak{D}^0(\Lambda_h^*)_0 = (\mathfrak{D}^0(\Lambda_h)_0)', \quad (3.12)$$

$$\mathfrak{R}(\Lambda_h^*)_0 = (\mathfrak{R}(\Lambda_h)_0)', \quad (3.13)$$

$$\mathfrak{R}^e(\Lambda_h^*)_0 = (\mathfrak{R}^e(\Lambda_h)_0)', \quad (3.14)$$

$$\mathfrak{R}^0(\Lambda_h^*)_0 = (\mathfrak{R}^0(\Lambda_h)_0)'. \quad (3.15)$$

*Proof. Step 1.* Given  $\rho \in ID_0$ , assume (3.8). Then,  $\mu \in ID_0$ . In order to prove (3.9), it is enough to show that

$$\int_0^{c_*} ds \int_{\mathbb{R}^d} (|f_*(s)x|^2 \wedge 1) \nu_{\rho'}(dx) < \infty, \quad (3.16)$$

$$\nu_{\mu'}(B) = \int_0^{c_*} ds \int_{\mathbb{R}^d} 1_B(f_*(s)x) \nu_{\rho'}(dx) \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \quad (3.17)$$

$$\gamma_{\mu'} = \int_{0+}^{c_*-} f_*(s) ds \left[ \gamma_{\rho'} + \int_{\mathbb{R}^d} x (1_{\{|f_*(s)x| \leq 1\}} - 1_{\{|x| \leq 1\}}) \nu_{\rho'}(dx) \right] \quad (3.18)$$

(see Theorems 3.5 and 3.10 of [25] or Proposition 3.18 of [26]). It follows from (3.8) that (3.16)–(3.18) hold for  $\mu$ ,  $\rho$ ,  $f(s)$ , and  $c$  in place of  $\mu'$ ,  $\rho'$ ,  $f_*(s)$ , and  $c_*$ . Thus

$$\nu_{\mu}(B) = \int_a^b h(t) dt \int_{\mathbb{R}^d} 1_B(tx) \nu_{\rho}(dx).$$

Using (1.1), we have

$$\begin{aligned} \nu_{\mu'}(B) &= \int_a^b h(t) dt \int_{\mathbb{R}^d} 1_B(t^{-1}|x|^{-2}x) t^2 |x|^2 \nu_{\rho}(dx) \\ &= \int_a^b h(t) dt \int_{\mathbb{R}^d} 1_B(t^{-1}x) t^2 \nu_{\rho'}(dx) = \int_{a_*}^{b_*} h^*(u) du \int_{\mathbb{R}^d} 1_B(ux) \nu_{\rho'}(dx). \end{aligned}$$

Hence (3.17) is true. We have (3.16) from (3.17), since  $\int (|x|^2 \wedge 1) \nu_{\mu'}(dx) < \infty$ . Moreover,

$$\int_{|x|=1} \nu_{\mu}(dx) = \int_0^c ds \int_{\mathbb{R}^d} 1_{\{f(s)|x|=1\}} \nu_{\rho}(dx) = \int_{\mathbb{R}^d} \nu_{\rho}(dx) \int_0^c 1_{\{f(s)=|x|^{-1}\}} ds = 0,$$

as  $f(s)$  is strictly decreasing. Hence, from (1.2) and from (3.18) for  $\mu$ ,

$$\gamma_{\mu'} = -\gamma_{\mu} = -\int_{0+}^{c-} f(s) ds \left[ \gamma_{\rho} + \int_{\mathbb{R}^d} x(1_{\{f(s)|x| \leq 1\}} - 1_{\{|x| \leq 1\}}) \nu_{\rho}(dx) \right].$$

Hence

$$\begin{aligned} \gamma_{\mu'} &= \int_{0+}^{c-} f(s) ds \left[ \gamma_{\rho'} - \int_{|x|=1} x \nu_{\rho'}(dx) - \int_{\mathbb{R}^d} x(1_{\{f(s)|x|^{-1} \leq 1\}} - 1_{\{|x|^{-1} \leq 1\}}) \nu_{\rho'}(dx) \right] \\ &= \int_{0+}^{c-} f(s) ds \left[ \gamma_{\rho'} + \int_{\mathbb{R}^d} x(1_{\{|x| > 1\}} - 1_{\{|x| \geq f(s)\}}) \nu_{\rho'}(dx) \right] \\ &= \int_{a+}^{b-} t h(t) dt \left[ \gamma_{\rho'} + \int_{\mathbb{R}^d} x(1_{\{|x| > 1\}} - 1_{\{|x| \geq t\}}) \nu_{\rho'}(dx) \right] \\ &= \int_{a_*+}^{b_*-} u h^*(u) du \left[ \gamma_{\rho'} + \int_{\mathbb{R}^d} x(1_{\{|x| > 1\}} - 1_{\{|x| \geq u^{-1}\}}) \nu_{\rho'}(dx) \right] \\ &= \int_{0+}^{c_*-} f_*(s) ds \left[ \gamma_{\rho'} + \int_{\mathbb{R}^d} x(1_{\{|x| > 1\}} - 1_{\{f_*(s)|x| \geq 1\}}) \nu_{\rho'}(dx) \right] \\ &= \int_{0+}^{c_*-} f_*(s) ds \left[ \gamma_{\rho'} + \int_{\mathbb{R}^d} x(1_{\{f_*(s)|x| < 1\}} - 1_{\{|x| \leq 1\}}) \nu_{\rho'}(dx) \right]. \end{aligned}$$

Since

$$\int_{0+}^{c_*-} f_*(s) ds \int_{\mathbb{R}^d} |x| 1_{\{f_*(s)|x|=1\}} \nu_{\rho'}(dx) = \int_{\mathbb{R}^d} |x| \nu_{\rho'}(dx) \int_{0+}^{c_*-} f_*(s) 1_{\{f_*(s)=|x|^{-1}\}} ds = 0$$

as  $f_*(s)$  is strictly decreasing, we obtain (3.18). Thus (3.9) holds. That is, (3.8) implies (3.9). Now (3.9) implies (3.8) automatically, since we have  $\rho'' = \rho$ ,  $\mu'' = \mu$ , and Proposition 3.3. We also obtain (3.10) and (3.13).

*Step 2.* Let us prove (3.11) and (3.14). Assume that  $\rho \in \mathfrak{D}^e(\Lambda_h)_0$ . Let  $k$  and  $Y$  be those in the definition of essential definability. Let  $\mu = \mathcal{L}(Y)$ . Then  $\mu \in ID_0$  and the analogue of (3.17) for  $\mu$  holds. As in Step 1, we obtain (3.16) and (3.17). Hence, by Theorem 3.6 of [25] or Proposition 3.18 of [26],  $\rho' \in \mathfrak{D}^e(\Lambda_h^*)_0$ . If  $h$  satisfies (C<sub>2</sub>) [resp. (C<sub>1</sub>)], then  $h^*$  satisfies (C<sub>1</sub>) [resp. (C<sub>2</sub>)], and we obtain  $\mu' \in \mathfrak{R}^e(\Lambda_h^*)$  from (3.17) and Proposition 3.27 of [26] [resp. an analogue of Proposition 3.27 of [26] for the  $\int_{0+}^c$  type integral in  $ID_0$ ]. Thus (3.11) and (3.14) are proved with  $=$  replaced by  $\supset$ . The converse inclusions automatically follow from this and (ii) of Proposition 2.1. Hence (3.11) and (3.14) are true.

*Step 3.* Let us prove (3.12) and (3.15). Assume that  $\rho \in \mathfrak{D}^0(\Lambda_h)_0$ . Then, by Proposition 2.3 of [24] or Proposition 3.18 of [26],

$$\int_0^c f(s)ds \left| \gamma_\rho + \int_{\mathbb{R}^d} x(1_{\{|f(s)|x| \leq 1\}} - 1_{\{|x| \leq 1\}}) \nu_\rho(dx) \right| < \infty.$$

The outer integral equals

$$\int_0^{c^*} f_*(s)ds \left| \gamma_{\rho'} + \int_{\mathbb{R}^d} x(1_{\{|f_*(s)|x| \leq 1\}} - 1_{\{|x| \leq 1\}}) \nu_{\rho'}(dx) \right|$$

by the same calculation as in Step 1. Since we already have (3.16), this shows that  $\rho' \in \mathfrak{D}^0(\Lambda_h^*)_0$ . Let  $\mu = \Lambda_h \rho$ . Then  $\mu' = \Lambda_h^* \rho'$  by the result of Step 1. Hence  $\mu' \in \mathfrak{R}^0(\Lambda_h^*)$ . Hence (3.12) and (3.15) are proved with  $=$  replaced by  $\supset$ . Then the converse inclusions are automatic.  $\square$

In view of Theorem 3.6, the relations of the domains and the ranges of  $\Phi_f$  with their restrictions to  $ID_0$  are of interest.

**Proposition 3.7.** *Let  $\Phi_f$  be a stochastic integral mapping. Then the classes  $\mathfrak{D}$ ,  $\mathfrak{D}^e$ ,  $\mathfrak{D}^0$ ,  $\mathfrak{R}$ ,  $\mathfrak{R}^e$ , and  $\mathfrak{R}^0$  of  $\Phi_f$  are closed under convolution.*

*Proof.* For  $\mathfrak{D}$  and  $\mathfrak{R}$  the assertion follows from the fact that if  $\rho_1, \rho_2 \in \mathfrak{D}(\Phi_f)$ , then  $\rho_1 * \rho_2 \in \mathfrak{D}(\Phi_f)$  and  $\Phi_f(\rho_1 * \rho_2) = (\Phi_f \rho_1) * (\Phi_f \rho_2)$ . It is in Propositions 3.18 and 3.20 of [26] and their analogue for improper stochastic integrals of  $\int_{0+}^c$  type in Section 3 of [25]. The other assertions are derived similarly.  $\square$

Let  $\mathfrak{S}_2 = \{\rho \in ID : 2\text{-stable}\} = \{\rho \in ID : \text{Gaussian}\}$  and let  $\mathfrak{S}_2^0 = \{\rho \in \mathfrak{S}_2 : m_\rho = 0\}$ .

**Proposition 3.8.** *Let  $\Phi_f$  be as in (1) and (2) in Section 1.*

(i) *If  $0 < \int_0^c f(s)^2 ds < \infty$ , then*

$$\begin{aligned} \mathfrak{D}(\Phi_f) &= \{\rho_1 * \rho_0 : \rho_1 \in \mathfrak{S}_2^0, \rho_0 \in \mathfrak{D}(\Phi_f)_0\}, \\ \mathfrak{D}^e(\Phi_f) &= \{\rho_1 * \rho_0 : \rho_1 \in \mathfrak{S}_2^0, \rho_0 \in \mathfrak{D}^e(\Phi_f)_0\}, \\ \mathfrak{D}^0(\Phi_f) &= \{\rho_1 * \rho_0 : \rho_1 \in \mathfrak{S}_2^0, \rho_0 \in \mathfrak{D}^0(\Phi_f)_0\}, \\ \mathfrak{R}(\Phi_f) &= \{\mu_1 * \mu_0 : \mu_1 \in \mathfrak{S}_2^0, \mu_0 \in \mathfrak{R}(\Phi_f)_0\}, \\ \mathfrak{R}^e(\Phi_f) &= \{\mu_1 * \mu_0 : \mu_1 \in \mathfrak{S}_2^0, \mu_0 \in \mathfrak{R}^e(\Phi_f)_0\}, \\ \mathfrak{R}^0(\Phi_f) &= \{\mu_1 * \mu_0 : \mu_1 \in \mathfrak{S}_2^0, \mu_0 \in \mathfrak{R}^0(\Phi_f)_0\}. \end{aligned}$$

(ii) *If  $\int_0^c f(s)^2 ds = \infty$ , then  $\mathfrak{D}$ ,  $\mathfrak{D}^e$ ,  $\mathfrak{D}^0$ ,  $\mathfrak{R}$ ,  $\mathfrak{R}^e$ , and  $\mathfrak{R}^0$  of  $\Phi_f$  are subclasses of  $ID_0$ .*

*Proof.* Use Proposition 3.18 of [26] and their analogue for improper stochastic integrals of  $\int_{0+}^c$  type, and note Proposition 3.7 above.  $\square$

#### 4. DOMAINS AND RANGES OF SOME STOCHASTIC INTEGRAL MAPPINGS AND THEIR CONJUGATES

We tackle the problem to find explicit description of the domains and the ranges of the stochastic integral mappings  $\bar{\Phi}_{p,\alpha}$ ,  $\Lambda_{q,\alpha}$ ,  $\Psi_{\alpha,\beta}$ , and their conjugates.

1.  $\bar{\Phi}_{p,\alpha}$  and its conjugate. Given  $p > 0$  and  $-\infty < \alpha < 2$ , let  $a = 0$ ,  $b = 1$ , and  $h(u) = \Gamma(p)^{-1}(1-u)^{p-1}u^{-\alpha-1}$ . Then  $h$  satisfies (C<sub>1</sub>). We have  $c = \Gamma(|\alpha|)/\Gamma(p+|\alpha|)$  if  $\alpha < 0$ , and  $c = \infty$  if  $\alpha \geq 0$ . The mapping  $\Lambda_h$  is denoted by  $\bar{\Phi}_{p,\alpha}$ , as in [26]. It is extensively studied in [23] in the notation  $\Phi_{\beta,\alpha} = \bar{\Phi}_{\beta-\alpha,\alpha}$ , and in [26]. The classes  $\mathfrak{R}(\Lambda_h)$ ,  $\mathfrak{R}^e(\Lambda_h)$ , and  $\mathfrak{R}^0(\Lambda_h)$  are denoted in [26] by  $K_{p,\alpha}$ ,  $K_{p,\alpha}^e$ , and  $K_{p,\alpha}^0$ , respectively. We have, as  $s \rightarrow \infty$ ,

$$f(s) \sim \begin{cases} \exp[C - \Gamma(p)s] & \text{if } \alpha = 0 \\ (\alpha\Gamma(p)s)^{-1/\alpha} & \text{if } 0 < \alpha < 2 \end{cases} \quad (4.1)$$

with a real constant  $C$  depending on  $p$ . If  $\alpha = 1$ , then the following more precise estimate is needed in the analysis of the domain as in Theorem 4.4 of [26]:

$$f(s) = (\Gamma(p)s)^{-1} - (1-p)(\Gamma(p)s)^{-2} \log s + O(s^{-2}). \quad (4.2)$$

We have  $a_* = 1$ ,  $b_* = \infty$ , and

$$h^*(u) = \Gamma(p)^{-1}(1-u^{-1})^{p-1}u^{\alpha+1-4} = \Gamma(p)^{-1}(u-1)^{p-1}u^{\alpha-p-2},$$

which satisfies (C<sub>2</sub>). Thus

$$g_*(t) = \Gamma(p)^{-1} \int_t^\infty (u-1)^{p-1}u^{\alpha-p-2}du, \quad t \in (1, \infty),$$

$$c_* = g_*(1+) = \Gamma(2-\alpha)/\Gamma(p+2-\alpha),$$

and  $f_*(s)$  for  $s \in (0, c_*)$  is the inverse function of  $g_*$ . For all  $\alpha \in (-\infty, 2)$ ,

$$\begin{aligned} g_*(t) &\sim (2-\alpha)^{-1}\Gamma(p)^{-1}t^{-(2-\alpha)}, & t \rightarrow \infty, \\ f_*(s) &\sim ((2-\alpha)\Gamma(p)s)^{-1/(2-\alpha)}, & s \downarrow 0. \end{aligned} \quad (4.3)$$

Notice that  $\int_0^{c_*} f_*(s)^2 ds < \infty$  if and only if  $\alpha < 0$ . If  $p = 1$ , then  $f$  and  $f_*$  are explicit, namely,  $h(u) = u^{-\alpha-1}$ ,

$$g(t) = \begin{cases} (1/\alpha)(t^{-\alpha} - 1) & \text{if } \alpha \neq 0 \\ -\log t & \text{if } \alpha = 0, \end{cases} \quad c = \begin{cases} |\alpha|^{-1} & \text{if } \alpha < 0 \\ \infty & \text{if } 0 \leq \alpha < 2, \end{cases}$$

$$f(s) = \begin{cases} (1-|\alpha|s)^{1/|\alpha|} & \text{if } \alpha < 0 \\ e^{-s} & \text{if } \alpha = 0 \\ (1+\alpha s)^{-1/\alpha} & \text{if } 0 < \alpha < 2, \end{cases}$$

$h^*(u) = u^{\alpha-3}$  for  $u \in (1, \infty)$  and

$$g_*(t) = (2-\alpha)^{-1}t^{-(2-\alpha)}, \quad c_* = (2-\alpha)^{-1},$$

$$f_*(s) = ((2 - \alpha)s)^{-1/(2-\alpha)} \quad \text{for } s \in (0, (2 - \alpha)^{-1}).$$

The mapping  $\bar{\Phi}_{1,\alpha}$  was studied by [9, 10, 12, 16]. If  $\alpha = -1$ , then  $f$  is explicit again:  $g(t) = \Gamma(p+1)^{-1}(1-t)^p$ ,  $c = \Gamma(p+1)^{-1}$ , and  $f(s) = 1 - (\Gamma(p+1)s)^{1/p}$ .

In order to describe the ranges, we need two definitions and a proposition.

**Definition 4.1.** ([26], p.7) Let  $p > 0$ . A  $[0, \infty]$ -valued function  $\varphi(u)$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ = (0, \infty)$ ] is said to be *monotone of order  $p$  on  $\mathbb{R}$*  [resp.  $\mathbb{R}_+^\circ$ ] if  $\varphi(u)$  is locally integrable on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ] and there is a locally finite measure  $\sigma$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ] such that

$$\varphi(u) = \Gamma(p)^{-1} \int_{(u, \infty)} (r - u)^{p-1} \sigma(dr) \quad \text{for } u \in \mathbb{R} \text{ [resp. } \mathbb{R}_+^\circ].$$

A function  $\varphi(u)$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ] is said to be *completely monotone on  $\mathbb{R}$*  [resp.  $\mathbb{R}_+^\circ$ ] if it is monotone of order  $p$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ] for all  $p > 0$ .

**Definition 4.2.** Let  $p > 0$ . A  $[0, \infty]$ -valued function  $\varphi(u)$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ] is said to be *increasing of order  $p$  on  $\mathbb{R}$*  [resp.  $\mathbb{R}_+^\circ$ ] if  $\varphi(u)$  is locally integrable on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ] and there is a locally finite measure  $\sigma$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ] such that

$$\begin{aligned} \varphi(u) &= \Gamma(p)^{-1} \int_{(-\infty, u)} (u - r)^{p-1} \sigma(dr) \quad \text{for } u \in \mathbb{R} \\ \text{[resp. } \varphi(u) &= \Gamma(p)^{-1} \int_{(0, u)} (u - r)^{p-1} \sigma(dr) \quad \text{for } u \in \mathbb{R}_+^\circ]. \end{aligned}$$

A function  $\varphi(u)$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ] is said to be *completely increasing on  $\mathbb{R}$*  [resp.  $\mathbb{R}_+^\circ$ ] if it is increasing of order  $p$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ] for all  $p > 0$ .

**Proposition 4.3.** Let  $p > 0$ . Let  $\varphi(u)$  be a  $[0, \infty]$ -valued function on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ]. Then  $\varphi(u)$  is increasing of order  $p$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ] if and only if  $\varphi(-u)$  [resp.  $u^{p-1}\varphi(u^{-1})$ ] is monotone of order  $p$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ]. In other words,  $\varphi(u)$  is monotone of order  $p$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ] if and only if  $\varphi(-u)$  [resp.  $u^{p-1}\varphi(u^{-1})$ ] is increasing of order  $p$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ].

*Proof.* If  $\varphi(u)$  is increasing of order  $p$  on  $\mathbb{R}$ , then  $\varphi(-u)$  is monotone of order  $p$  on  $\mathbb{R}$ , and conversely, since

$$\varphi(-u) = \Gamma(p)^{-1} \int_{(-\infty, -u)} (-u - r)^{p-1} \sigma(dr) = \Gamma(p)^{-1} \int_{(u, \infty)} (r - u)^{p-1} \sigma^-(dr)$$

with  $\sigma^-(B) = \sigma(-B)$  for  $B \in \mathcal{B}(\mathbb{R})$ . If  $\varphi(u)$  is increasing of order  $p$  on  $\mathbb{R}_+^\circ$ , then  $u^{p-1}\varphi(u^{-1})$  is monotone of order  $p$  on  $\mathbb{R}_+^\circ$ , and conversely, since

$$\begin{aligned} u^{p-1}\varphi(u^{-1}) &= \Gamma(p)^{-1} u^{p-1} \int_{(0, u^{-1})} (u^{-1} - r)^{p-1} \sigma(dr) \\ &= \Gamma(p)^{-1} \int_{(u, \infty)} (r - u)^{p-1} r^{1-p} \sigma^{-1}(dr) \end{aligned}$$



with  $\sigma^{-1}(B) = \int_{\mathbb{R}_+^\circ} 1_B(r^{-1})\sigma(dr)$  for  $B \in \mathcal{B}(\mathbb{R}_+^\circ)$ . Note that, on  $\mathbb{R}_+^\circ$ ,  $\psi(u) = u^{p-1}\varphi(u^{-1})$  if and only if  $\varphi(u) = u^{p-1}\psi(u^{-1})$ .  $\square$

*Remark.* Assume that  $p$  is a positive integer and that  $\varphi(u)$  is  $p$  times differentiable. Then  $\varphi(u)$  is monotone of order  $p$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ] if and only if  $(-d/du)^n\varphi(u) \geq 0$  for  $n = 1, 2, \dots, p$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ] and  $\varphi(u) \rightarrow 0$  as  $u \rightarrow \infty$ ;  $\varphi(u)$  is increasing of order  $p$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ] if and only if  $(d/du)^n\varphi \geq 0$  for  $n = 1, 2, \dots, p$  on  $\mathbb{R}$  [resp.  $\mathbb{R}_+^\circ$ ] and  $\varphi(u) \rightarrow 0$  as  $u \rightarrow -\infty$  [resp.  $u \rightarrow 0$ ]. See Corollary 2.12 of [26] for the proof of the first assertion. The proof of the second assertion is given by a modification of that of the first.

A function  $\varphi(u)$  is completely increasing on  $\mathbb{R}$  if and only if  $\varphi(-u)$  is completely monotone on  $\mathbb{R}$ . If a function  $\varphi(u)$  is completely increasing on  $\mathbb{R}_+^\circ$ , then  $\varphi(u^{-1})$  is completely monotone on  $\mathbb{R}_+^\circ$ . However, the converse of the last statement is not true; consider  $\varphi(u) = u^\alpha$  with  $\alpha$  being positive and non-integer.

We will also use the concepts for  $\mu \in ID$  to have weak mean  $m_\mu$  and to have weak mean  $m_\mu$  absolutely, introduced in [26]. We say that  $\mu \in ID$  has weak mean  $m_\mu$  if  $\int_{1 < |x| \leq a} x\nu_\mu(dx)$  is convergent in  $\mathbb{R}^d$  as  $a \rightarrow \infty$  and if

$$\widehat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, A_\mu z \rangle + \lim_{a \rightarrow \infty} \int_{|x| \leq a} (e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle)\nu_\mu(dx) + i\langle m_\mu, z \rangle\right], \quad z \in \mathbb{R}^d.$$

If  $\mu \in ID$  has weak mean  $m_\mu$ , then we have  $m_\mu = \gamma_\mu + \lim_{a \rightarrow \infty} \int_{1 < |x| \leq a} x\nu_\mu(dx)$ . We say that  $\mu \in ID$  has weak mean  $m_\mu$  absolutely if  $\mu$  has weak mean  $m_\mu$  and if  $\int_{(1, \infty)} r\bar{\nu}_\mu(dr) \left| \int_S \xi \lambda_r^\mu(d\xi) \right| < \infty$ , where  $(\bar{\nu}_\mu(dr), \lambda_r^\mu(d\xi))$  is a spherical decomposition of  $\nu_\mu$ . This property is independent of the choice of spherical decompositions of  $\nu_\mu$ . If  $\mu \in ID$  has mean  $m_\mu$ , then  $\mu$  has weak mean  $m_\mu$  absolutely.

Now let us give description of the domains and the ranges of  $\bar{\Phi}_{p,\alpha}$  and its conjugate. The results on  $\bar{\Phi}_{p,\alpha}$  are already known; our emphasis lies on the counterpart in the results on their conjugates.

**Theorem 4.4.** *Let  $p > 0$  and  $-\infty < \alpha < 2$ . Let  $\Lambda_h = \bar{\Phi}_{p,\alpha}$ .*

(i) *The domains and the ranges in the essentially definable sense are as follows:*

$$\mathfrak{D}^e(\Lambda_h) = \begin{cases} ID & \text{if } \alpha < 0 \\ \{\rho \in ID : \int_{|x| > 1} \log |x| \nu_\rho(dx) < \infty\} & \text{if } \alpha = 0 \\ \{\rho \in ID : \int_{|x| > 1} |x|^\alpha \nu_\rho(dx) < \infty\} & \text{if } 0 < \alpha < 2, \end{cases} \quad (4.4)$$

$$\begin{aligned} \mathfrak{R}^e(\Lambda_h) = \{ \mu \in ID : \nu_\mu \text{ has a rad. dec. } (\lambda(d\xi), u^{-\alpha-1}k_\xi(u)du) \text{ such that } k_\xi(u) \\ \text{is measurable in } (\xi, u) \text{ and monotone of order } p \text{ in } u \in \mathbb{R}_+^\circ \} \quad \text{for all } \alpha, \end{aligned} \quad (4.5)$$

$$\mathfrak{D}^e(\Lambda_h^*) = \begin{cases} ID & \text{if } \alpha < 0 \\ \{\rho \in ID_0 : \int_{|x|<1} (-\log|x|)|x|^2 \nu_\rho(dx) < \infty\} & \text{if } \alpha = 0 \\ \{\rho \in ID_0 : \int_{|x|<1} |x|^{2-\alpha} \nu_\rho(dx) < \infty\} & \text{if } 0 < \alpha < 2, \end{cases} \quad (4.6)$$

$$\mathfrak{R}^e(\Lambda_h^*)_0 = \{\mu \in ID_0 : \nu_\mu \text{ has a rad. dec. } (\lambda(d\xi), u^{\alpha-p-2} k_\xi(u) du) \text{ such that } k_\xi(u) \text{ is measurable in } (\xi, u) \text{ and increasing of order } p \text{ in } u \in \mathbb{R}_+^\circ\} \text{ for all } \alpha, \quad (4.7)$$

$$\mathfrak{R}^e(\Lambda_h^*) = \begin{cases} \{\mu_1 * \mu_0 : \mu_1 \in \mathfrak{S}_2^0, \mu_0 \in \mathfrak{R}^e(\Lambda_h^*)_0\} & \text{if } \alpha < 0 \\ \mathfrak{R}^e(\Lambda_h^*)_0 & \text{if } 0 \leq \alpha < 2. \end{cases} \quad (4.8)$$

(ii) If  $-\infty < \alpha < 1$ , then

$$\mathfrak{D}(\Lambda_h) = \mathfrak{D}^0(\Lambda_h) = \mathfrak{D}^e(\Lambda_h), \quad (4.9)$$

$$\mathfrak{R}(\Lambda_h) = \mathfrak{R}^0(\Lambda_h) = \mathfrak{R}^e(\Lambda_h), \quad (4.10)$$

$$\mathfrak{D}(\Lambda_h^*) = \mathfrak{D}^0(\Lambda_h^*) = \mathfrak{D}^e(\Lambda_h^*), \quad (4.11)$$

$$\mathfrak{R}(\Lambda_h^*) = \mathfrak{R}^0(\Lambda_h^*) = \mathfrak{R}^e(\Lambda_h^*). \quad (4.12)$$

(iii) If  $\alpha = 1$ , then

$$\mathfrak{D}(\Lambda_h) = \{\rho \in \mathfrak{D}^e(\Lambda_h) : m_\rho = 0, \lim_{r \rightarrow \infty} \int_1^r s^{-1} ds \int_{|x|>s} x \nu_\rho(dx) \text{ exists in } \mathbb{R}^d\}, \quad (4.13)$$

$$\mathfrak{D}^0(\Lambda_h) = \{\rho \in \mathfrak{D}^e(\Lambda_h) : m_\rho = 0, \int_1^\infty s^{-1} ds \left| \int_{|x|>s} x \nu_\rho(dx) \right| < \infty\}, \quad (4.14)$$

$$\mathfrak{R}(\Lambda_h) = \{\mu \in \mathfrak{R}^e(\Lambda_h) : \mu \text{ has weak mean } 0\}, \quad (4.15)$$

$$\mathfrak{R}^0(\Lambda_h) = \{\mu \in \mathfrak{R}^e(\Lambda_h) : \mu \text{ has weak mean } 0 \text{ absolutely}\}, \quad (4.16)$$

$$\mathfrak{D}(\Lambda_h^*) = \{\rho \in \mathfrak{D}^e(\Lambda_h^*) : \gamma_\rho^0 = 0, \lim_{r \downarrow 0} \int_r^1 s^{-1} ds \int_{|x|<s} x \nu_\rho(dx) \text{ exists in } \mathbb{R}^d\}, \quad (4.17)$$

$$\mathfrak{D}^0(\Lambda_h^*) = \{\rho \in \mathfrak{D}^e(\Lambda_h^*) : \gamma_\rho^0 = 0, \int_0^1 s^{-1} ds \left| \int_{|x|<s} x \nu_\rho(dx) \right| < \infty\}. \quad (4.18)$$

(iv) If  $1 < \alpha < 2$ , then

$$\mathfrak{D}(\Lambda_h) = \mathfrak{D}^0(\Lambda_h) = \{\rho \in \mathfrak{D}^e(\Lambda_h) : \rho \text{ has mean } 0\}, \quad (4.19)$$

$$\mathfrak{R}(\Lambda_h) = \mathfrak{R}^0(\Lambda_h) = \{\mu \in \mathfrak{R}^e(\Lambda_h) : \mu \text{ has mean } 0\}, \quad (4.20)$$

$$\mathfrak{D}(\Lambda_h^*) = \mathfrak{D}^0(\Lambda_h^*) = \{\rho \in \mathfrak{D}^e(\Lambda_h^*) : \rho \text{ has drift } 0\}, \quad (4.21)$$

$$\mathfrak{R}(\Lambda_h^*) = \mathfrak{R}^0(\Lambda_h^*) = \{\mu \in \mathfrak{R}^e(\Lambda_h^*) : \mu \text{ has drift } 0\}. \quad (4.22)$$

*Remark.* The expression (4.7) can be replaced by the following:

$$\mathfrak{R}^e(\Lambda_h^*)_0 = \{\mu \in ID_0 : \nu_\mu \text{ has a rad. dec. } (\lambda(d\xi), u^{\alpha-3} k_\xi(u^{-1}) du) \text{ such that } k_\xi(v) \text{ is measurable in } (\xi, v) \text{ and monotone of order } p \text{ in } v \in \mathbb{R}_+^\circ\} \text{ for all } \alpha. \quad (4.23)$$

*Remark.* Description of  $\mathfrak{R}(\Lambda_h^*)$  and  $\mathfrak{R}^0(\Lambda_h^*)$  in case  $\alpha = 1$  for  $\Lambda_h = \bar{\Phi}_{p,\alpha}$  will be given in another paper [28].

*Proof of Theorem 4.4.* All assertions concerning  $\Lambda_h$  are known; see [26] (Theorems 4.2, 4.15, 4.18, and 4.21) and also [23] together with (4.1) and (4.2). Then, applying Theorem 3.6, we obtain all results on the domains and the ranges of  $\Lambda_h^*$  intersected with  $ID_0$ . Thereafter use Proposition 3.8 to remove the restriction to  $ID_0$ , recalling that  $\int_0^c f(s)^2 ds < \infty$  for all  $\alpha$  and that  $\int_0^{c*} f_*(s)^2 ds < \infty$  if and only if  $\alpha < 0$ , from (4.1) and (4.3). The details of the proof of (4.7), (4.17), and (4.18) are as follows.

Let us show (4.7). Assume  $\mu \in \mathfrak{R}^e(\Lambda_h)_0$ . Then  $\mu' \in \mathfrak{R}^e(\Lambda_h^*)_0$  from Theorem 3.6 and, since

$$\nu_\mu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(u\xi) u^{-\alpha-1} k_\xi(u) du$$

with  $k_\xi(u)$  monotone of order  $p$  in  $u \in \mathbb{R}_+^\circ$ , we have, from (1.1),

$$\begin{aligned} \nu_{\mu'}(B) &= \int_S \lambda(d\xi) \int_0^\infty 1_B(u^{-1}\xi) u^{1-\alpha} k_\xi(u) du = \int_S \lambda(d\xi) \int_0^\infty 1_B(v\xi) v^{\alpha-3} k_\xi(v^{-1}) dv \\ &= \int_S \lambda(d\xi) \int_0^\infty 1_B(v\xi) v^{\alpha-p-2} v^{p-1} k_\xi(v^{-1}) dv. \end{aligned}$$

Hence, exchanging the roles of  $\mu$  and  $\mu'$ , we see from Proposition 4.3 that  $\mathfrak{R}^e(\Lambda_h^*)_0$  is a subclass of the right-hand side of (4.7). Similarly we can prove that  $\mathfrak{R}^e(\Lambda_h)_0$  includes the right-hand side of (4.7).

To prove (4.17) and (4.18), notice that  $m_\rho = -\gamma_{\rho'}^0$ ,

$$\begin{aligned} \int_1^r s^{-1} ds \int_{|x|>s} x \nu_\rho(dx) &= \int_1^r s^{-1} ds \int_{|x|^{-1}>s} |x|^{-2} x |x|^2 \nu_{\rho'}(dx) \\ &= \int_1^r s^{-1} ds \int_{|x|<s^{-1}} x \nu_{\rho'}(dx) = \int_{r^{-1}}^1 s^{-1} ds \int_{|x|<s} x \nu_{\rho'}(dx), \end{aligned}$$

and similarly

$$\int_1^r s^{-1} ds \left| \int_{|x|>s} x \nu_\rho(dx) \right| = \int_{r^{-1}}^1 s^{-1} ds \left| \int_{|x|<s} x \nu_{\rho'}(dx) \right|,$$

and apply Theorem 3.6.  $\square$

2.  $\Lambda_{q,\alpha}$  and its conjugate. Given  $q > 0$  and  $-\infty < \alpha < 2$ , let  $a = 0$ ,  $b = 1$ , and  $h(u) = \Gamma(q)^{-1}(-\log u)^{q-1} u^{-\alpha-1}$ . Then  $h$  satisfies (C<sub>1</sub>). We have  $c = |\alpha|^{-q}$  if  $\alpha < 0$ , and  $c = \infty$  if  $\alpha \geq 0$ . The mapping  $\Lambda_h$  is denoted by  $\Lambda_{q,\alpha}$ , as in [26]; there it is extensively studied. The classes  $\mathfrak{R}(\Lambda_h)$ ,  $\mathfrak{R}^e(\Lambda_h)$ , and  $\mathfrak{R}^0(\Lambda_h)$  are denoted by  $L_{q,\alpha}$ ,  $L_{q,\alpha}^e$ , and  $L_{q,\alpha}^0$ , respectively. It is known that  $\Lambda_{q_1+q_2,\alpha} = \Lambda_{q_2,\alpha} \Lambda_{q_1,\alpha}$  for  $\alpha \neq 1$ . We have

$$f(s) = \exp[-(\Gamma(q+1)s)^{1/q}] \quad \text{for } s \in (0, \infty) \quad \text{if } \alpha = 0, \quad (4.24)$$

$$f(s) \sim (\alpha \Gamma(q)s)^{-1/\alpha} (\alpha^{-1} \log s)^{(q-1)/\alpha}, \quad s \rightarrow \infty, \quad \text{if } \alpha > 0 \quad (4.25)$$

(Proposition 6.1 of [26]). We have  $a_* = 1$ ,  $b_* = \infty$ , and

$$h^*(u) = \Gamma(q)^{-1}(\log u)^{q-1} u^{\alpha-3},$$

which satisfies (C<sub>2</sub>). Thus

$$g_*(t) = \Gamma(q)^{-1} \int_{\log t}^{\infty} v^{q-1} e^{-(2-\alpha)v} dv, \quad t \in (1, \infty),$$

$$c_* = g_*(1+) = (2-\alpha)^{-q},$$

and  $f_*(s)$  for  $s \in (0, c_*)$  is the inverse function of  $g_*$ . For all  $\alpha \in (-\infty, 2)$  we have

$$g_*(t) \sim (2-\alpha)^{-1} \Gamma(q)^{-1} t^{-(2-\alpha)} (\log t)^{q-1}, \quad t \rightarrow \infty.$$

It follows from this that

$$f_*(s) \sim (2-\alpha)^{-q/(2-\alpha)} (\Gamma(q)s)^{-1/(2-\alpha)} (-\log s)^{(q-1)/(2-\alpha)}, \quad s \downarrow 0. \quad (4.26)$$

The proof is left to the reader. Again notice that  $\int_0^{c_*} f_*(s)^2 ds < \infty$  if and only if  $\alpha < 0$ . If  $q = 1$ , then  $\Lambda_{1,\alpha} = \bar{\Phi}_{1,\alpha}$ . If  $\alpha = 0$ , then  $\Lambda_{q,0}$  with  $q = 1, 2, \dots$  coincides with the mapping introduced by [7].

**Theorem 4.5.** *Let  $q > 0$  and  $-\infty < \alpha < 2$ . Let  $\Lambda_h = \Lambda_{q,\alpha}$ .*

(i) *The domains and the ranges in the essentially definable sense are as follows:*

$$\mathfrak{D}^e(\Lambda_h) = \begin{cases} ID & \text{if } \alpha < 0 \\ \{\rho \in ID: \int_{|x|>1} (\log |x|)^q \nu_\rho(dx) < \infty\} & \text{if } \alpha = 0 \\ \{\rho \in ID: \int_{|x|>2} (\log |x|)^{q-1} |x|^\alpha \nu_\rho(dx) < \infty\} & \text{if } 0 < \alpha < 2, \end{cases} \quad (4.27)$$

$$\mathfrak{R}^e(\Lambda_h) = \{\mu \in ID: \nu_\mu \text{ has a rad. dec. } (\lambda(d\xi), u^{-\alpha-1} h_\xi(\log u) du) \text{ such that } h_\xi(y) \text{ is measurable in } (\xi, y) \text{ and monotone of order } q \text{ in } y \in \mathbb{R}\} \quad \text{for all } \alpha, \quad (4.28)$$

$$\mathfrak{D}^e(\Lambda_h^*) = \begin{cases} ID & \text{if } \alpha < 0 \\ \{\rho \in ID_0: \int_{|x|<1} (-\log |x|)^q |x|^2 \nu_\rho(dx) < \infty\} & \text{if } \alpha = 0 \\ \{\rho \in ID_0: \int_{|x|<1/2} (-\log |x|)^{q-1} |x|^{2-\alpha} \nu_\rho(dx) < \infty\} & \text{if } 0 < \alpha < 2, \end{cases} \quad (4.29)$$

$$\mathfrak{R}^e(\Lambda_h^*)_0 = \{\mu \in ID_0: \nu_\mu \text{ has a rad. dec. } (\lambda(d\xi), u^{\alpha-3} h_\xi(\log u) du) \text{ such that } h_\xi(y) \text{ is measurable in } (\xi, y) \text{ and increasing of order } q \text{ in } y \in \mathbb{R}\} \quad \text{for all } \alpha, \quad (4.30)$$

$$\mathfrak{R}^e(\Lambda_h^*) = \begin{cases} \{\mu_1 * \mu_0: \mu_1 \in \mathfrak{S}_2^0, \mu_0 \in \mathfrak{R}^e(\Lambda_h^*)_0\} & \text{if } \alpha < 0 \\ \mathfrak{R}^e(\Lambda_h^*)_0 & \text{if } 0 \leq \alpha < 2. \end{cases} \quad (4.31)$$

(ii) *If  $-\infty < \alpha < 1$ , then we have the same assertion as in (ii) of Theorem 4.4.*

(iii) *If  $\alpha = 1$  and  $q \geq 1$ , then*

$$\mathfrak{D}(\Lambda_h) = \{\rho \in \mathfrak{D}^e(\Lambda_h): m_\rho = 0, \lim_{r \rightarrow \infty} \int_1^r (\log s)^{q-1} s^{-1} ds \int_{|x|>s} x \nu_\rho(dx) \text{ exists in } \mathbb{R}^d\}, \quad (4.32)$$

$$\mathfrak{D}^0(\Lambda_h) = \{\rho \in \mathfrak{D}^e(\Lambda_h): m_\rho = 0, \int_1^\infty (\log s)^{q-1} s^{-1} ds \left| \int_{|x|>s} x \nu_\rho(dx) \right| < \infty\}, \quad (4.33)$$

$$\mathfrak{D}(\Lambda_h^*) = \{\rho \in \mathfrak{D}^e(\Lambda_h^*) : \gamma_\rho^0 = 0, \lim_{r \downarrow 0} \int_r^1 (-\log s)^{q-1} s^{-1} ds \int_{|x| < s} x \nu_\rho(dx) \text{ exists in } \mathbb{R}^d\}, \quad (4.34)$$

$$\mathfrak{D}^0(\Lambda_h^*) = \{\rho \in \mathfrak{D}^e(\Lambda_h^*) : \gamma_\rho^0 = 0, \int_0^1 (-\log s)^{q-1} s^{-1} ds \left| \int_{|x| < s} x \nu_\rho(dx) \right| < \infty\}. \quad (4.35)$$

(iv) If  $1 < \alpha < 2$ , then we have the same assertion as in (iv) of Theorem 4.4.

*Remark.* In the case  $\Lambda_h = \Lambda_{q,\alpha}$ , we do not know how to describe  $\mathfrak{D}(\Lambda_h)_0$ ,  $\mathfrak{D}^0(\Lambda_h)_0$ ,  $\mathfrak{D}(\Lambda_h^*)_0$ , and  $\mathfrak{D}^0(\Lambda_h^*)_0$  for  $\alpha = 1$  and  $0 < q < 1$ , and  $\mathfrak{R}(\Lambda_h)_0$ ,  $\mathfrak{R}^0(\Lambda_h)_0$ ,  $\mathfrak{R}(\Lambda_h^*)_0$ , and  $\mathfrak{R}^0(\Lambda_h^*)_0$  for  $\alpha = 1$  and  $q \neq 1$ . If  $\alpha = 1$  and  $q = 1$ , then  $\Lambda_{1,1} = \bar{\Phi}_{1,1}$  and Theorem 4.4 applies.

*Proof of Theorem 4.5.* This is proved similarly to Theorem 4.4. That is, start from the fact that all assertions concerning  $\Lambda_h$  are known in [26] (Theorems 6.2, 6.3, 6.9, and 6.12). Notice that  $\int_0^c f(s)^2 ds < \infty$  for all  $\alpha$  and that  $\int_0^{c*} f_*(s)^2 ds < \infty$  if and only if  $\alpha < 0$ , from (4.24)–(4.26).  $\square$

3.  $\Psi_{\alpha,\beta}$  and its conjugate. Given  $-\infty < \alpha < 2$  and  $\beta > 0$ , let  $a = 0$ ,  $b = \infty$ , and  $h(u) = u^{-\alpha-1} e^{-u^\beta}$ . Then  $h$  satisfies (C<sub>1</sub>) and  $g(t) = \beta^{-1} \int_{t^\beta}^\infty v^{-\alpha\beta^{-1}-1} e^{-v} dv$ . We have  $c = \beta^{-1} \Gamma(|\alpha| \beta^{-1})$  if  $\alpha < 0$ , and  $c = \infty$  if  $\alpha \geq 0$ . The mapping  $\Lambda_h$  is denoted by  $\Psi_{\alpha,\beta}$  as in [13, 17]. As  $t \downarrow 0$ ,

$$g(t) = \begin{cases} -\log t + C + o(1) & \text{with some } C \in \mathbb{R} \text{ if } \alpha = 0 \\ \alpha^{-1} t^{-\alpha} (1 + o(1)) & \text{if } \alpha > 0. \end{cases}$$

Hence, as  $s \rightarrow \infty$ ,

$$f(s) \sim \begin{cases} e^{C-s} & \text{if } \alpha = 0 \\ (\alpha s)^{-1/\alpha} & \text{if } \alpha > 0. \end{cases} \quad (4.36)$$

If  $\alpha = 1$ , then more precisely

$$g(t) = \begin{cases} t^{-1} + O(1) & \text{if } \beta > 1 \\ t^{-1} + \log t + O(1) & \text{if } \beta = 1 \\ t^{-1} - (1 - \beta)^{-1} t^{\beta-1} (1 + o(1)) & \text{if } 0 < \beta < 1. \end{cases}$$

as  $t \downarrow 0$  and it follows that

$$f(s) = \begin{cases} s^{-1} + O(s^{-2}) & \text{if } \beta > 1 \\ s^{-1} - s^{-2} \log s + O(s^{-2}) & \text{if } \beta = 1 \\ s^{-1} + O(s^{-\beta-1}) & \text{if } 0 < \beta < 1 \end{cases} \quad (4.37)$$

as  $s \rightarrow \infty$ . We have  $a_* = 0$ ,  $b_* = \infty$ , and  $h^*(u) = u^{\alpha-3} e^{-u^{-\beta}}$ , which satisfies (C<sub>2</sub>). Thus,  $g_*(t) = \beta^{-1} \int_0^{t^{-\beta}} v^{(2-\alpha)\beta^{-1}-1} e^{-v} dv$  for  $t \in (0, \infty)$  and  $c_* = \beta^{-1} \Gamma((2-\alpha)\beta^{-1})$ ;  $f_*(s)$  for  $s \in (0, c_*)$  is the inverse function of  $g_*$ . We have, for all  $\alpha$  and  $\beta$ ,

$$g_*(t) \sim (2-\alpha)^{-1} t^{-(2-\alpha)}, \quad t \rightarrow \infty,$$

$$f_*(s) \sim ((2 - \alpha)s)^{-1/(2-\alpha)}, \quad s \downarrow 0. \quad (4.38)$$

Letting  $g_{\alpha,\beta}$  and  $f_{\alpha,\beta}$  denote the function  $g$  and  $f$ , respectively, for given  $\alpha$  and  $\beta$ , we have  $g_{\alpha,\beta}(t) = \beta^{-1}g_{\alpha/\beta,1}(t^\beta)$  and  $f_{\alpha,\beta}(s) = f_{\alpha/\beta,1}(\beta s)^{1/\beta}$ . If  $\beta = 1$ , then  $\Psi_{\alpha,1}$  equals  $\Psi_\alpha$  studied in [23, 26]. In [26] the classes  $\mathfrak{R}(\Psi_{\alpha,1})$ ,  $\mathfrak{R}^e(\Psi_{\alpha,1})$ , and  $\mathfrak{R}^0(\Psi_{\alpha,1})$  are shown to be equal to  $K_{\infty,\alpha} = \bigcap_{p>0} K_{p,\alpha}$ ,  $K_{\infty,\alpha}^e = \bigcap_{p>0} K_{p,\alpha}^e$ , and  $K_{\infty,\alpha}^0 = \bigcap_{p>0} K_{p,\alpha}^0$ , respectively. If  $\beta = 1$  and  $\alpha = -1$ , then  $f(s) = -\log s$  for  $0 < s < 1$  and  $\Psi_{-1,1}$  equals  $\Upsilon$  studied in [4, 6]. If  $\alpha = -\beta$ , then  $f(s) = (-\log \beta s)^{1/\beta}$  for  $0 < s < 1/\beta$  and  $\Psi_{-\beta,\beta}$  was treated in [1]. If  $\alpha = 0$  and  $\beta = 2$ ,  $\Psi_{0,2}$  was studied in [3].

**Theorem 4.6.** *Let  $-\infty < \alpha < 2$  and  $\beta > 0$ . Let  $\Lambda_h = \Psi_{\alpha,\beta}$ .*

(i) *The domains and the ranges in the essentially definable sense are as follows:*

$$\mathfrak{D}^e(\Lambda_h) = \begin{cases} ID & \text{if } \alpha < 0 \\ \{\rho \in ID: \int_{|x|>1} \log |x| \nu_\rho(dx) < \infty\} & \text{if } \alpha = 0 \\ \{\rho \in ID: \int_{|x|>1} |x|^\alpha \nu_\rho(dx) < \infty\} & \text{if } 0 < \alpha < 2, \end{cases} \quad (4.39)$$

$$\mathfrak{R}^e(\Lambda_h) = \{\mu \in ID: \nu_\mu \text{ has a rad. dec. } (\lambda(d\xi), u^{-\alpha-1} k_\xi(u^\beta) du) \text{ such that } k_\xi(v) \text{ is measurable in } (\xi, v) \text{ and completely monotone in } v \in \mathbb{R}_+^\circ\} \quad \text{for all } \alpha, \quad (4.40)$$

$$\mathfrak{D}^e(\Lambda_h^*) = \begin{cases} ID & \text{if } \alpha < 0 \\ \{\rho \in ID_0: \int_{|x|<1} (-\log |x|) |x|^2 \nu_\rho(dx) < \infty\} & \text{if } \alpha = 0 \\ \{\rho \in ID_0: \int_{|x|<1} |x|^{2-\alpha} \nu_\rho(dx) < \infty\} & \text{if } 0 < \alpha < 2, \end{cases} \quad (4.41)$$

$$\mathfrak{R}^e(\Lambda_h^*)_0 = \{\mu \in ID_0: \nu_\mu \text{ has a rad. dec. } (\lambda(d\xi), u^{\alpha-3} k_\xi(u^{-\beta}) du) \text{ such that } k_\xi(v) \text{ is measurable in } (\xi, v) \text{ and completely monotone in } v \in \mathbb{R}_+^\circ\} \quad \text{for all } \alpha, \quad (4.42)$$

$$\mathfrak{R}^e(\Lambda_h^*) = \begin{cases} \{\mu_1 * \mu_0: \mu_1 \in \mathfrak{S}_2^0, \mu_0 \in \mathfrak{R}^e(\Lambda_h^*)_0\} & \text{if } \alpha < 0 \\ \mathfrak{R}^e(\Lambda_h^*)_0 & \text{if } 0 \leq \alpha < 2. \end{cases} \quad (4.43)$$

(ii) *Concerning  $\mathfrak{D}$ ,  $\mathfrak{D}^0$ ,  $\mathfrak{R}$ , and  $\mathfrak{R}^0$  of  $\Lambda_h$  and  $\Lambda_h^*$ , the statements in (ii), (iii), and (iv) of Theorem 4.4 are true word by word.*

*Remark.* Description of  $\mathfrak{R}(\Lambda_h^*)$  and  $\mathfrak{R}^0(\Lambda_h^*)$  in case  $\alpha = 1$  for  $\Lambda_h = \Psi_{\alpha,\beta}$  will be given in another paper [28].

*Proof of Theorem 4.6.* Concerning the domains of  $\Lambda_h$  and  $\Lambda_h^*$ , the proof is the same as Theorem 4.4. The ranges of  $\Lambda_h$  are given in [23] and [26] (Theorems 5.8 and 5.10) in case  $\beta = 1$  and treated in Theorem 2.8 of [13] for  $\alpha < 1$  and  $\beta \neq 1$ . To deal with  $\mathfrak{R}(\Lambda_h)$  and  $\mathfrak{R}^0(\Lambda_h)$  in case  $1 \leq \alpha < 2$  with  $\beta \neq 1$ , we can extend the method of the proof of Theorem 5.10 of [26]. The assertions on the ranges of  $\Lambda_h^*$  are proved from the assertions on  $\Lambda_h$  in the same way as in the proof of Theorem 4.4.  $\square$

Maejima and Ueda [16] introduced the class  $L^{(\alpha)}$  and showed its connection to  $\mathfrak{R}^e(\Lambda_h)$  with  $\Lambda_h = \Lambda_{1,\alpha} = \bar{\Phi}_{1,\alpha}$ . In the rest of this section we introduce a class  $L^{(\alpha)*}$  and study its connection to  $\mathfrak{R}^e(\Lambda_h^*)$  with  $\Lambda_h = \Lambda_{1,\alpha} = \bar{\Phi}_{1,\alpha}$ . The definition of  $L^{(\alpha)}$  in [16] is as follows. A distribution  $\mu \in ID$  is called  $\alpha$ -selfdecomposable with  $\alpha \in \mathbb{R}$  if, for any  $b > 1$ , there is  $\rho_b \in ID$  satisfying

$$\hat{\mu}(z) = \hat{\mu}(b^{-1}z)^{b^\alpha} \hat{\rho}_b(z), \quad z \in \mathbb{R}^d. \quad (4.44)$$

The totality of  $\alpha$ -selfdecomposable distributions on  $\mathbb{R}^d$  is denoted by  $L^{(\alpha)}$ . The 0-selfdecomposability coincides with the selfdecomposability. The paper [16] treated distributions in  $L^{(\alpha)}$  systematically. Earlier this class was studied by Jurek [9, 10] and others in 1970s and 80s and also in [12]; see references in [16]. The following properties of  $L^{(\alpha)}$  are known.

- (i)  $\mu \in L^{(\alpha)}$  if and only if, for any  $b > 1$ ,  $A_\mu - b^{\alpha-2}A_\mu$  is nonnegative-definite and  $\nu_\mu \geq b^\alpha T_{b^{-1}} \nu_\mu$ .
- (ii) If  $\mu_1, \mu_2 \in L^{(\alpha)}$ , then  $\mu_1 * \mu_2 \in L^{(\alpha)}$ .
- (iii) If  $\mu \in L^{(\alpha)}$ , then  $\mu^s \in L^{(\alpha)}$  for  $s > 0$ .
- (iv) If  $\alpha_1 < \alpha_2$ , then  $L^{(\alpha_1)} \supset L^{(\alpha_2)}$ .
- (v)  $\bigcap_{\beta < \alpha} L^{(\beta)} = L^{(\alpha)}$ .
- (vi) If  $\alpha > 2$ , then  $L^{(\alpha)} = \{\delta_c : c \in \mathbb{R}^d\}$ .
- (vii)  $L^{(2)} = \mathfrak{S}_2$ , the class of 2-stable (that is, Gaussian) distributions.
- (viii) If  $0 < \alpha \leq \beta < 2$ , then  $L^{(\alpha)}$  contains all  $\beta$ -stable distributions.
- (ix) If  $0 < \beta < \alpha < 2$ , then  $L^{(\alpha)}$  does not contain any non-trivial  $\beta$ -stable distribution.
- (x) Let  $-\infty < \alpha < 2$ . Then  $\mu \in L^{(\alpha)}$  if and only if  $\nu_\mu$  has a radial decomposition  $(\lambda(d\xi), u^{-\alpha-1}k_\xi(u)du)$  such that  $k_\xi(u)$  is measurable in  $(\xi, u)$  and, for  $\lambda$ -a. e.  $\xi$ , decreasing on  $\mathbb{R}_+^\circ$  in  $u$ .
- (xi) If  $\alpha \leq 0$ , then  $L^{(\alpha)} = \mathfrak{R}^e(\Lambda_{1,\alpha})$ .
- (xii) If  $0 < \alpha < 2$ , then  $\mu \in L^{(\alpha)}$  if and only if  $\mu = \mu_0 * \mu_1$  where  $\mu_0 \in \mathfrak{R}^e(\Lambda_{1,\alpha})$  and  $\mu_1$  is  $\alpha$ -stable.

**Definition 4.7.** Let  $\alpha \in \mathbb{R}$ . The class  $L^{(\alpha)*}$  is the totality of  $\mu \in ID$  such that, for any  $b > 1$ , there is  $\sigma_b \in ID$  satisfying

$$\hat{\mu}(z) = \hat{\mu}(bz)^{b^{\alpha-2}} \hat{\sigma}_b(z), \quad z \in \mathbb{R}^d. \quad (4.45)$$

**Proposition 4.8.** Let  $\alpha \in \mathbb{R}$ . The class  $L^{(\alpha)*}$  has the following properties.

- (i)  $\mu \in L^{(\alpha)*}$  if and only if, for any  $b > 1$ ,  $A_\mu - b^\alpha A_\mu$  is nonnegative-definite and  $\nu_\mu \geq b^{\alpha-2} T_b \nu_\mu$ .
- (ii) If  $\mu_1, \mu_2 \in L^{(\alpha)*}$ , then  $\mu_1 * \mu_2 \in L^{(\alpha)*}$ .
- (iii) If  $\mu \in L^{(\alpha)*}$ , then  $\mu^s \in L^{(\alpha)*}$  for  $s > 0$ .
- (iv) If  $\alpha_1 < \alpha_2$ , then  $L^{(\alpha_1)*} \supset L^{(\alpha_2)*}$ .

- (v)  $\bigcap_{\beta < \alpha} L^{(\beta)*} = L^{(\alpha)*}$ .
- (vi) If  $\alpha \geq 2$ , then  $L^{(\alpha)*} = \{\delta_c : c \in \mathbb{R}^d\}$ .
- (vii) If  $\alpha > 0$ , then  $L^{(\alpha)*} \subset ID_0$ .
- (viii)  $L^{(0)*} \supset \mathfrak{S}_2$ .
- (ix) If  $0 < \alpha \leq \beta < 2$ , then  $L^{(\alpha)*}$  contains all  $(2 - \beta)$ -stable distributions.
- (x) If  $0 < \beta < \alpha < 2$ , then  $L^{(\alpha)*}$  does not contain any non-trivial  $(2 - \beta)$ -stable distribution.

Proof is straightforward.

**Theorem 4.9.** Let  $\alpha \in \mathbb{R}$  and let  $\mu \in ID_0$ . Then  $\mu \in L^{(\alpha)}$  if and only if  $\mu' \in L^{(\alpha)*}$ .

*Proof.* Assume that  $\mu \in L^{(\alpha)}$ . Then  $\mu = (T_{b^{-1}}\mu)^{b^\alpha} * \rho_b$ . Using Proposition 2.4, we obtain  $\mu' = ((T_{b^{-1}}\mu)')^{b^\alpha} * \rho'_b = (T_b(\mu'))^{b^{\alpha-2}} * \rho'_b$ . Hence  $\mu' \in L^{(\alpha)*}$  with  $\sigma_b = \rho'_b$ . In a similar way we can show that  $\mu' \in L^{(\alpha)*}$  implies  $\mu \in L^{(\alpha)}$ .  $\square$

Notice that, for  $\alpha = 0$ , Theorem 4.9 gives a characterization of the inversions of selfdecomposable distributions in  $ID_0$  in terms of a new kind of decomposability.

**Proposition 4.10.** Let  $-\infty < \alpha < 2$  and  $\mu \in ID_0$ . Then  $\mu \in L^{(\alpha)*}$  if and only if  $\nu_\mu$  has a radial decomposition  $(\lambda(d\xi), u^{\alpha-3}k_\xi(u)du)$  such that  $k_\xi(u)$  is measurable in  $(\xi, u)$  and, for  $\lambda$ -a. e.  $\xi$ , increasing in  $u \in \mathbb{R}_+^\circ$ .

*Proof.* Using Theorem 4.9 and property (x) of  $L^{(\alpha)}$ , we can prove the assertion similarly to the proof of (4.7).  $\square$

**Proposition 4.11.** If  $\alpha < 0$ , then  $L^{(\alpha)*} = \mathfrak{R}^e(\Lambda_{1,\alpha}^*)$ . If  $0 \leq \alpha < 2$ , then  $\mu \in L^{(\alpha)*}$  if and only if  $\mu = \mu_0 * \mu_1$  where  $\mu_0 \in \mathfrak{R}^e(\Lambda_{1,\alpha}^*) = \mathfrak{R}^e(\Lambda_{1,\alpha}^*)_0$  and  $\mu_1$  is  $(2 - \alpha)$ -stable.

*Proof.* Combine properties (xi) and (xii) of  $L^{(\alpha)}$  with (iii) of Theorem 2.5, (3.11) of Theorem 3.6, (i) of Theorem 4.5, and Theorem 4.9.  $\square$

## 5. SIMILAR RESULTS IN OTHER CASES

The definition of the conjugates of stochastic integral mappings  $\Phi_f$  is restricted to the case where  $f$  is a positive function satisfying some condition. We do not know how to define conjugates of general stochastic integral mappings. But we can obtain similar results in a case where  $f$  takes positive and negative values both.

**Definition 5.1.** A function  $h$  is said to satisfy *Condition (D)* if  $h$  is defined on  $\mathbb{R} \setminus \{0\}$ , positive, and measurable, and

$$\int_{\mathbb{R} \setminus \{0\}} h(u)(1 + u^2)du < \infty. \quad (5.1)$$

For any function  $h$  satisfying Condition (D), define  $h^*(u) = h(1/u)/u^4$  for  $u \in \mathbb{R} \setminus \{0\}$ .



**Proposition 5.2.** *If  $h$  satisfies Condition (D), then  $h^*$  satisfies Condition (D) and  $(h^*)^* = h$ .*

Proof is straightforward.

Let  $h$  be a function satisfying Condition (D). Let

$$g_h(t) = \int_{(t, \infty) \setminus \{0\}} h(u) du, \quad t \in \mathbb{R}. \quad (5.2)$$

Then  $g_h(t)$  is strictly decreasing continuous function with  $g_h(-\infty) < \infty$  and  $g_h(\infty) = 0$ . Let  $c_h = g_h(-\infty)$ . Hence  $c_h < \infty$ . Define  $f_h(s)$  as

$$s = g_h(t) \text{ with } -\infty < t < \infty \Leftrightarrow t = f_h(s) \text{ with } 0 < s < c_h.$$

Then  $f_h(s)$  is a strictly decreasing, continuous function with  $f_h(0+) = \infty$  and  $f_h(c_h-) = -\infty$ , and  $\int_0^{c_h} f_h(s)^2 ds = \int_{\mathbb{R} \setminus \{0\}} u^2 h(u) du < \infty$ . The improper stochastic integral  $\lim_{p \downarrow 0, q \uparrow c_h} \int_p^q f_h(s) dX_s^{(\rho)}$  is convergent in probability for all  $\rho \in ID$ , as is proved in Theorem 6.1 of [25]. The distribution of this improper stochastic integral is denoted by  $\Lambda_h \rho$ . We have  $\mathfrak{D}(\Lambda_h) = ID$ ; there is no need to consider essentially definable case. We have  $\int_0^{c_h} |\log \widehat{\rho}(f_h(s)z)| ds < \infty$  for all  $\rho \in ID$ , that is,  $\Lambda_h \rho$  is absolutely definable for all  $\rho \in ID$  (see Theorem 6.1 of [25]). It is easy to see that  $\Lambda_h \rho \in ID_0$  if and only if  $\rho \in ID_0$ .

**Definition 5.3.** If  $h$  satisfies Condition (D), then  $\Lambda_{h^*}$  is called the *conjugate* of  $\Lambda_h$ . Write  $\Lambda_h^* = \Lambda_{h^*}$ .

It follows from Proposition 5.2 that the conjugate of  $\Lambda_h^*$  is  $\Lambda_h$ .

**Theorem 5.4.** *Let  $h$  be a function satisfying Condition (D). Let  $\rho$  and  $\mu$  be in  $ID_0$ . Then  $\Lambda_h \rho = \mu$  if and only if  $\Lambda_h^* \rho' = \mu'$ . Consequently,  $\mathfrak{R}(\Lambda_h^*)_0 = (\mathfrak{R}(\Lambda_h)_0)'$ .*

*Proof.* We proceed as in the proof of Theorem 3.6. But this time we have to be careful as  $u$  and  $1/u$  are discontinuous at 0 and  $f_h(s)$  and  $f_{h^*}(s)$  take positive and negative values. Details are omitted.  $\square$

**Example 5.5.** Let  $h(u) = (2\pi)^{-1/2} e^{-u^2/2}$  for  $u \in \mathbb{R} \setminus \{0\}$ . Then  $h$  satisfies Condition (D) and  $h^*(u) = (2\pi)^{-1/2} e^{-1/(2u^2)} (1/u^4)$  for  $u \in \mathbb{R} \setminus \{0\}$  and  $c_h = 1$ . Let  $ID^{\text{sym}}$  denote the class of symmetric infinitely divisible distributions on  $\mathbb{R}^d$ . Then

$$\mathfrak{R}(\Lambda_h) = \{\mu \in ID^{\text{sym}} : \nu_\mu \text{ has a rad. dec. } (\lambda(d\xi), k_\xi(u^2)du) \text{ such that } k_\xi(v) \text{ is measurable in } (\xi, v) \text{ and, for } \lambda\text{-a.e. } \xi, \text{ completely monotone in } v \in \mathbb{R}_+^\circ\}.$$

This is essentially a result of [2, 14]. The class  $\mathfrak{R}(\Lambda_h)$  is identical with the class of type  $G$  distributions on  $\mathbb{R}^d$  of Maejima and Rosiński [14]. Then, Theorem 5.4 and a discussion similar to the proof of Theorem 4.6 show that

$$\mathfrak{R}(\Lambda_h^*)_0 = (\mathfrak{R}(\Lambda_h)_0)' = \{\mu \in ID_0^{\text{sym}} : \nu_\mu \text{ has a rad. dec. } (\lambda(d\xi), u^{-4} k_\xi(u^{-2})du) \text{ such that}$$

$k_\xi(v)$  is measurable in  $(\xi, v)$  and, for  $\lambda$ -a. e.  $\xi$ , completely monotone in  $v \in \mathbb{R}_+^\circ$ .

For  $\mu \in ID$  let  $\mu_0$  and  $\mu_1$  denote the infinitely divisible distributions with triplets  $(0, \nu_\mu, \gamma_\mu)$  and  $(A_\mu, 0, 0)$ , respectively. Then we have

$$\mathfrak{R}(\Lambda_h^*) = \{\mu \in ID^{\text{sym}} : \mu = \mu_1 * \mu_0 \text{ with } \mu_1 \in \mathfrak{S}_2^0, \mu_0 \in \mathfrak{R}(\Lambda_h^*)_0\}$$

similarly to Proposition 3.8, since  $0 < \int_0^{c_h^*} f_{h^*}(s)^2 ds < \infty$ .

## 6. LIMITS OF SOME NESTED CLASSES

Let us make a study of the limit of the ranges of the iterations of  $\Lambda_h$  and  $\Lambda_h^*$ . The iteration  $\Phi_f^n$  of a stochastic integral mapping  $\Phi_f$  is defined as  $\Phi_f^1 = \Phi_f$  and  $\Phi_f^{n+1}\rho = \Phi_f(\Phi_f^n\rho)$  with  $\mathfrak{D}(\Phi_f^{n+1}) = \{\rho \in \mathfrak{D}(\Phi_f^n) : \Phi_f^n\rho \in \mathfrak{D}(\Phi_f)\}$ . We have  $ID \supset \mathfrak{R}(\Phi_f) \supset \mathfrak{R}(\Phi_f^2) \supset \dots$ . The limit class is denoted by  $\mathfrak{R}_\infty(\Phi_f) = \bigcap_{n=1}^\infty \mathfrak{R}(\Phi_f^n)$ . In the case where  $\Lambda_h$  equals  $\bar{\Phi}_{p,\alpha}$ ,  $\Lambda_{q,\alpha}$ , or  $\Psi_{\alpha,\beta}$ , the description of  $\mathfrak{R}_\infty(\Lambda_h)$  is studied in [1, 15, 17, 18, 20, 26, 27]. In [27] it is obtained for  $\alpha \in (-\infty, 1) \cup (1, 2)$ ,  $p \geq 1$ ,  $q > 0$ , and  $\beta = 1$  and for  $\alpha = 1$ ,  $p \geq 1$ ,  $q = 1$ , and  $\beta = 1$ ; now we want to describe  $\mathfrak{R}_\infty(\Lambda_h^*)$ . We will see new classes appear as  $\mathfrak{R}_\infty(\Lambda_h^*)$  for some parameter values. We are also interested in finding what parameters are relevant. It will be shown that only the parameter  $\alpha$  is relevant and the parameters  $p$  and  $q$  are irrelevant.

We need the class  $L_\infty$  in the study of  $\mathfrak{R}_\infty(\Phi_f)$ . It is the class of completely selfdecomposable distributions on  $\mathbb{R}^d$ , which is the smallest class that is closed under convolution and weak convergence and contains all stable distributions on  $\mathbb{R}^d$ . A distribution  $\mu \in ID$  belongs to  $L_\infty$  if and only if  $\nu_\mu$  is represented as

$$\nu_\mu(B) = \int_{(0,2)} \Gamma_\mu(d\beta) \int_S \lambda_\beta^\mu(d\xi) \int_0^\infty 1_B(r\xi) r^{-\beta-1} dr, \quad B \in \mathcal{B}(\mathbb{R}^d), \quad (6.1)$$

where  $\Gamma_\mu$  is a measure on the open interval  $(0, 2)$  satisfying  $\int_{(0,2)} (\beta^{-1} + (2-\beta)^{-1}) \Gamma_\mu(d\beta) < \infty$  and  $\{\lambda_\beta^\mu : \beta \in (0, 2)\}$  is a measurable family of probability measures on  $S$ . This  $\Gamma_\mu$  is uniquely determined by  $\nu_\mu$  and  $\{\lambda_\beta^\mu\}$  is determined by  $\nu_\mu$  up to  $\beta$  of  $\Gamma_\mu$ -measure 0. For a Borel subset  $E$  of the interval  $(0, 2)$ , the class  $L_\infty^E$  denotes the totality of  $\mu \in L_\infty$  such that  $\Gamma_\mu$  is concentrated on  $E$ . The class  $L_\infty^{(\alpha,2)}$  for  $0 < \alpha < 2$  appears in [18, 26, 27] in the description of  $\mathfrak{R}_\infty(\Phi_f)$  for some  $f$ . We will use  $(L_\infty)_0 = L_\infty \cap ID_0$  and  $(L_\infty^E)_0 = L_\infty^E \cap ID_0$ .

**Proposition 6.1.** *Let  $E \in \mathcal{B}((0, 2))$ . Let  $2 - E$  denote the set  $\{2 - \beta : \beta \in E\}$ . Let  $\mu \in (L_\infty)_0$ . Then  $\mu \in L_\infty^E$  if and only if  $\mu' \in L_\infty^{2-E}$ .*

*Proof.* Assume that  $\mu \in L_\infty^E$ . Then  $\Gamma_\mu((0, 2) \setminus E) = 0$ . Define  $\Gamma^\sharp(F) = \int_{(0,2)} 1_F(2 - \beta) \Gamma_\mu(d\beta)$  for  $F \in \mathcal{B}((0, 2))$ . Then  $\int_{(0,2)} (\beta^{-1} + (2 - \beta)^{-1}) \Gamma^\sharp(d\beta) < \infty$  and  $\Gamma^\sharp((0, 2) \setminus (2 - E)) = 0$ . For  $B \in \mathcal{B}(\mathbb{R}^d)$  we have, from (1.1) and (6.1),

$$\nu_{\mu'}(B) = \int_E \Gamma_\mu(d\beta) \int_S \lambda_\beta^\mu(d\xi) \int_0^\infty 1_B(r^{-1}\xi) r^{-\beta+1} dr$$

$$\begin{aligned}
&= \int_E \Gamma_\mu(d\beta) \int_S \lambda_\beta^\mu(d\xi) \int_0^\infty 1_B(u\xi) u^{\beta-3} du \\
&= \int_{(0,2)} 1_{2-E}(\beta) \Gamma^\sharp(d\beta) \int_S \lambda_{2-\beta}^\mu(d\xi) \int_0^\infty 1_B(u\xi) u^{-1-\beta} du.
\end{aligned}$$

Therefore  $\mu' \in L_\infty^{2-E}$ . Now it is automatic that  $\mu' \in L_\infty^{2-E}$  implies  $\mu \in L_\infty^E$ , since  $\mu'' = \mu$ .  $\square$

**Proposition 6.2.** *Let  $h$  be a function satisfying Condition (C). Let  $n$  be a positive integer. Let  $\rho \in ID_0$ . Then*

$$\rho \in \mathfrak{D}(\Lambda_h^n) \quad \text{and} \quad \Lambda_h^n \rho = \mu$$

*if and only if*

$$\rho' \in \mathfrak{D}((\Lambda_h^*)^n) \quad \text{and} \quad (\Lambda_h^*)^n \rho' = \mu'.$$

*Thus,  $\mathfrak{D}((\Lambda_h^*)^n)_0 = (\mathfrak{D}(\Lambda_h^n)_0)'$  and  $\mathfrak{R}((\Lambda_h^*)^n)_0 = (\mathfrak{R}(\Lambda_h^n)_0)'$ .*

*Proof.* Using Theorem 3.6, we can show the assertion by induction in  $n$ .  $\square$

**Theorem 6.3.** *Let  $h$  be a function satisfying Condition (C). Consider  $\Lambda_h$  and  $\Lambda_h^*$ . Then*

$$\mathfrak{R}_\infty(\Lambda_h^*)_0 = (\mathfrak{R}_\infty(\Lambda_h)_0)'.$$

*Proof.* It follows from  $\mathfrak{R}((\Lambda_h^*)^n)_0 = (\mathfrak{R}(\Lambda_h^n)_0)'$  that

$$\mathfrak{R}_\infty(\Lambda_h^*)_0 = \bigcap_{n=1}^\infty \mathfrak{R}((\Lambda_h^*)^n)_0 = \bigcap_{n=1}^\infty (\mathfrak{R}(\Lambda_h^n)_0)' = \left( \bigcap_{n=1}^\infty \mathfrak{R}(\Lambda_h^n)_0 \right)' = (\mathfrak{R}_\infty(\Lambda_h)_0)',$$

completing the proof.  $\square$

**Theorem 6.4.** *Let  $\Lambda_h$  be one of  $\bar{\Phi}_{p,\alpha}$ ,  $\Lambda_{q,\alpha}$ , and  $\Psi_{\alpha,1}$ . The classes  $\mathfrak{R}_\infty(\Lambda_h)$  and  $\mathfrak{R}_\infty(\Lambda_h^*)$  are as follows.*

- (i) *If  $\alpha < 0$ ,  $p \geq 1$ , and  $q > 0$ , then  $\mathfrak{R}_\infty(\Lambda_h) = \mathfrak{R}_\infty(\Lambda_h^*) = L_\infty$ .*
- (ii) *If  $0 \leq \alpha < 1$ ,  $p \geq 1$ , and  $q > 0$ , then*

$$\mathfrak{R}_\infty(\Lambda_h) = L_\infty^{(\alpha,2)}, \quad \mathfrak{R}_\infty(\Lambda_h^*) = (L_\infty^{(0,2-\alpha)})_0.$$

- (iii) *If  $\alpha = 1$ ,  $p \geq 1$ , and  $q = 1$ , then*

$$\mathfrak{R}_\infty(\Lambda_h) = L_\infty^{(1,2)} \cap \{\mu \in ID : \mu \text{ has weak mean } 0\}.$$

- (iv) *If  $1 < \alpha < 2$ ,  $p \geq 1$ , and  $q > 0$ , then*

$$\begin{aligned}
\mathfrak{R}_\infty(\Lambda_h) &= L_\infty^{(\alpha,2)} \cap \{\mu \in ID : \mu \text{ has mean } 0\}, \\
\mathfrak{R}_\infty(\Lambda_h^*) &= (L_\infty^{(0,2-\alpha)})_0 \cap \{\mu \in ID_0 : \mu \text{ has drift } 0\}.
\end{aligned}$$

*Proof.* All results on  $\mathfrak{R}_\infty(\Lambda_h)$  are given in [27]. Hence, using Theorem 6.3, we see from Propositions 6.1 and 2.1 (v) that  $\mathfrak{R}_\infty(\Lambda_h^*)_0$  equals  $(L_\infty)_0$ ,  $(L_\infty^{(0,2-\alpha)})_0$ , or  $(L_\infty^{(0,2-\alpha)})_0 \cap \{\mu \in ID_0: \mu \text{ has drift } 0\}$  in (i), (ii), or (iv), respectively.

Let us prove (i). Note that  $0 < \int_0^{c_h^*} f_{h^*}(s)ds < \infty$ . In general, we can prove  $\mathfrak{R}(\Phi_f^n) = \{\mu_1 * \mu_0: \mu_1 \in \mathfrak{S}_2^0, \mu_0 \in \mathfrak{R}(\Phi_f^n)_0\}$  and  $\mathfrak{R}_\infty(\Phi_f) = \{\mu_1 * \mu_0: \mu_1 \in \mathfrak{S}_2^0, \mu_0 \in \mathfrak{R}_\infty(\Phi_f)_0\}$  in (i) of Proposition 3.8, repeating the same argument. Hence  $\mathfrak{R}_\infty(\Lambda_h^*) = L_\infty$ .

Proof of (ii) and (iv) is as follows. In this case we have  $\int_0^{c_h^*} f_{h^*}(s)^2 ds = \infty$ . Hence  $\mathfrak{R}_\infty(\Lambda_h^*) \subset ID_0$ , as in (ii) of Proposition 3.8.  $\square$

*Remark.* A supplement of Theorem 6.4 (iii) for  $\mathfrak{R}_\infty(\Lambda_h^*)$  will be given in [28]. Theorem 6.4 does not cover  $\Psi_{\alpha,\beta}$  with  $\beta \neq 1$ , because we rely on the results of [27]. However, using the result of [17], we can extend Theorem 6.4 to some of  $\Psi_{\alpha,\beta}$  with  $\beta \neq 1$ .

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